Solving multivariate quadratic systems in practice

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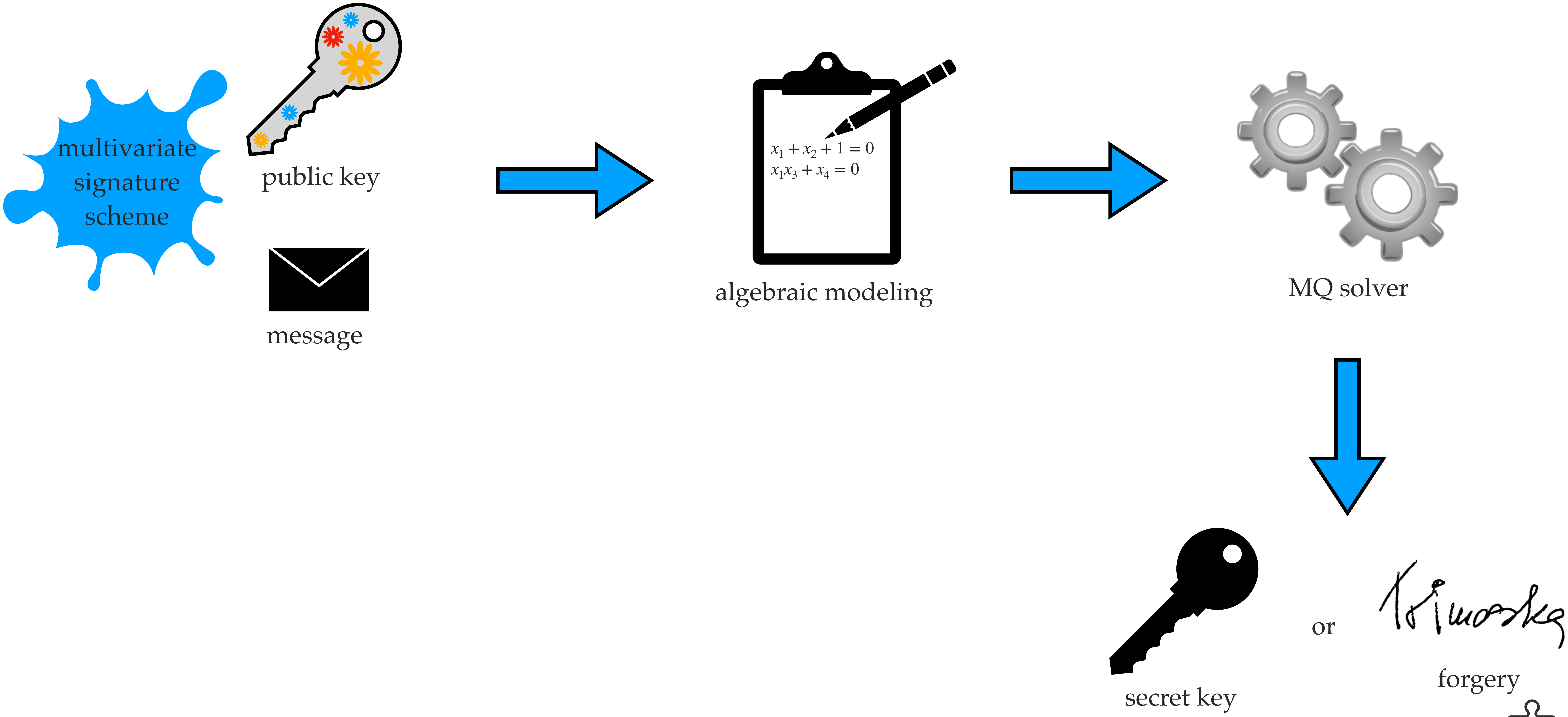
TU/e

*Animated version at: https://mtrimoska.com/slides/Croatia_solvers/

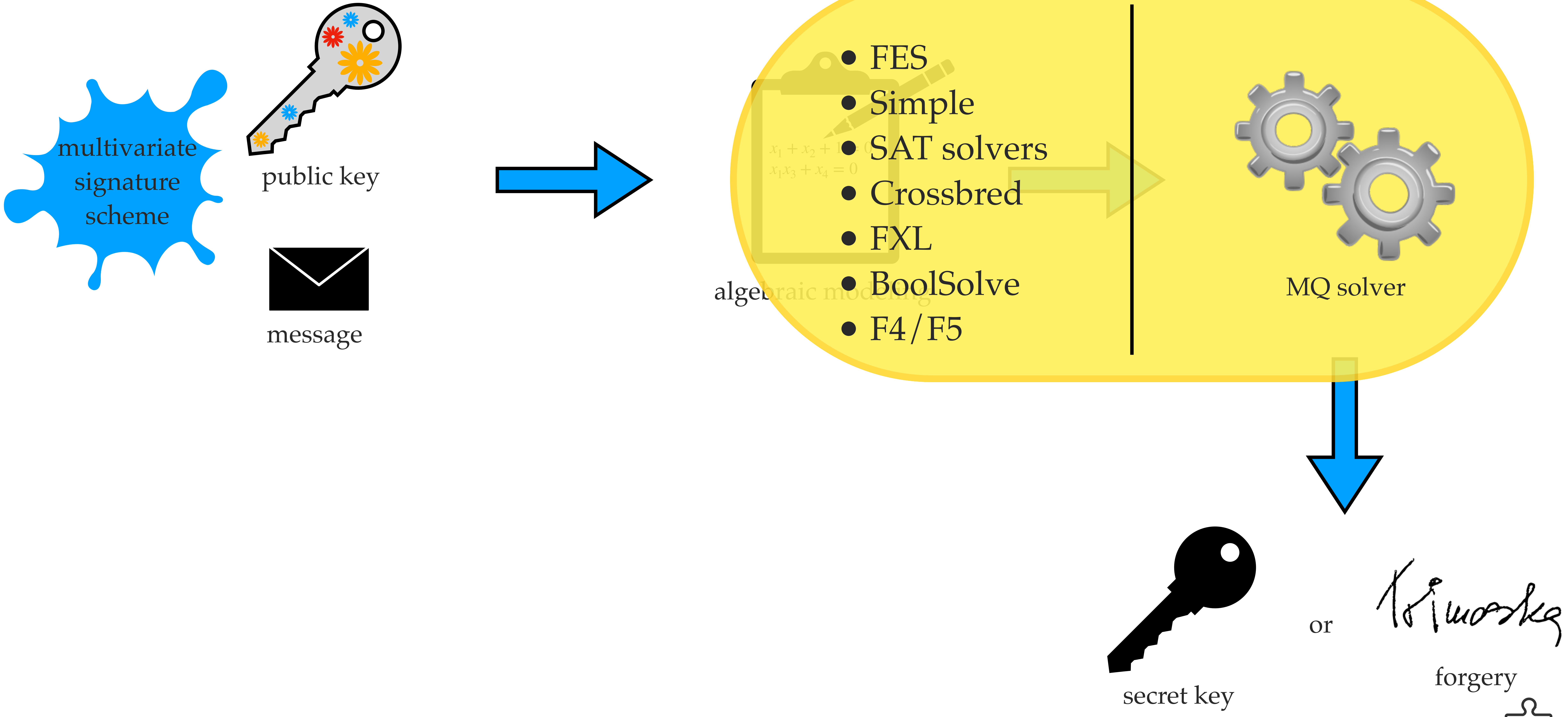
Algebraic cryptanalysis

A type of cryptanalytic methods where the problem of finding the secret key (or any attack goal) is reduced to the problem of finding a solution to a nonlinear multivariate polynomial system of equations.

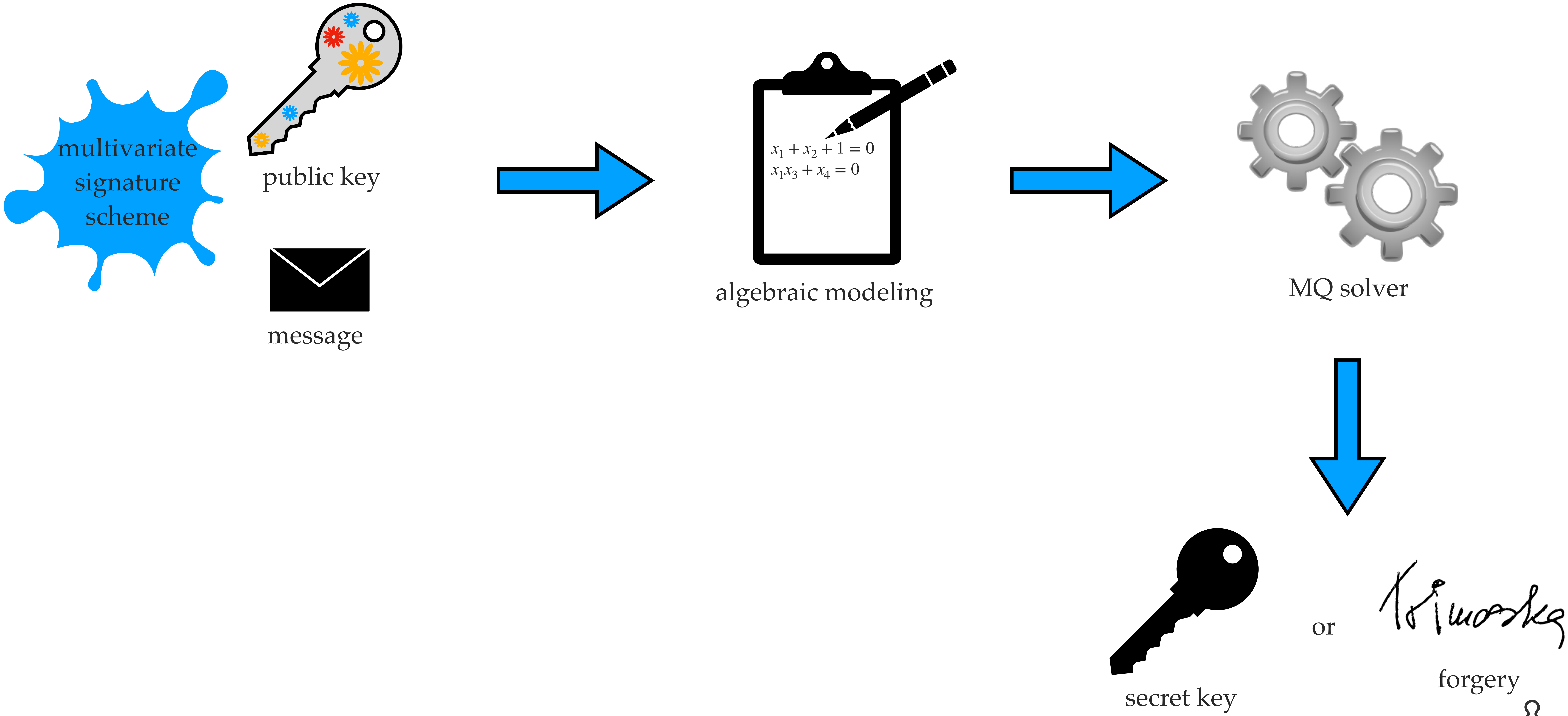
Algebraic cryptanalysis



Algebraic cryptanalysis

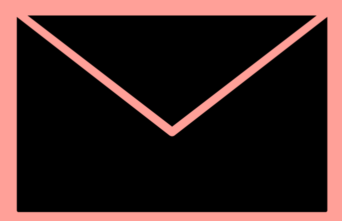


Algebraic cryptanalysis



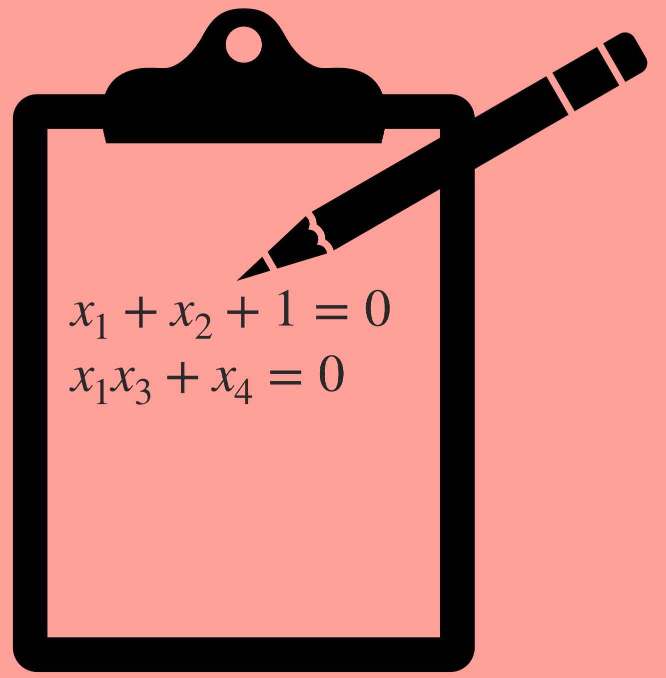
Algebraic cryptanalysis

public key



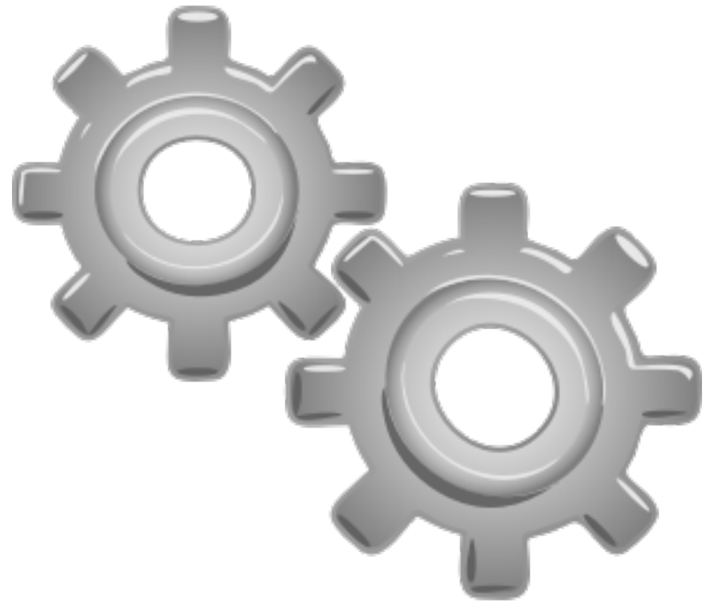
message

- UOV



algebraic modeling

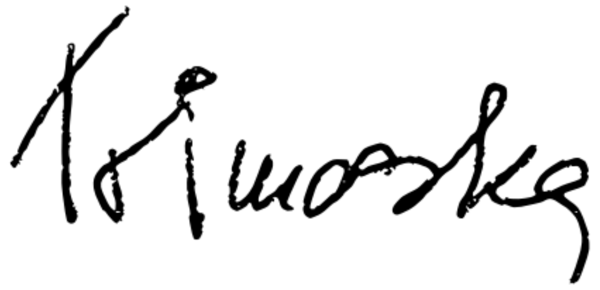
- Direct attack
- Kipnis-Shamir
- Reconciliation
- Intersection



MQ solver

secret key

or



forgery

The MQ problem (recall)

The MQ problem

Given m multivariate quadratic polynomials $f_1, \ldots, f_m$ of n variables over a finite field $\mathbb{F}_q$, find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f_1(\mathbf{x}) = \ldots = f_m(\mathbf{x}) = 0$.

Example.

$$
\begin{aligned}
f_1 &: x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0 \\
f_2 &: x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0 \\
f_3 &: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0 \\
f_4 &: x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0 \\
f_5 &: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0 \\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
$$

Algebraic cryptanalysis : modelisation example

Example.

Given matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2 \in \mathscr{M}_{n,n}(\mathbb{F}_q)$ (the space of matrices over $\mathbb{F}_q$ of size $n \times n$), find $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ (the space of invertible matrices over $\mathbb{F}_q$ of size $n \times n$), such that

$$\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$$
$$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$$
$$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$$

$$\underbrace{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} \end{bmatrix}}_{\mathbf{D}_1} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & c_{3,5} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & c_{4,5} \\ c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & c_{5,5} \end{bmatrix}}_{\mathbf{C}_1} \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & b_{5,5} \end{bmatrix}}_{\mathbf{B}}$$

Algebraic cryptanalysis : modelisation example

Example. $\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$

$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$

$$\mathbf{D}_1 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} \end{bmatrix} = \mathbf{A}\mathbf{C}_1\mathbf{B} = \begin{bmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,5} \end{bmatrix}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$$
$$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$$

$$d_{1,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} = 0,$$

$$\underbrace{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} \end{bmatrix}}_{\mathbf{D}_1} = \underbrace{\begin{bmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,5} \end{bmatrix}}_{\mathbf{A}\mathbf{C}_1\mathbf{B}}$$

Algebraic cryptanalysis : modelisation example

Example.

$$\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$$
$$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$$

$$d_{1,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} = 0,$$

$$d_{2,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} = 0,$$

$$\begin{array}{c} \mathbf{D}_1 \\ \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} \end{bmatrix} \end{array} = \begin{array}{c} \mathbf{A}\,\mathbf{C}_1\mathbf{B} \\ \begin{bmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,5} \end{bmatrix} \end{array}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$$
$$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$$

$$d_{1,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} = 0, \qquad d_{1,2} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2} = 0, \ldots$$

$$d_{2,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} = 0,$$

$\mathbf{D}_1$

$d_{1,1}$	$d_{1,2}$	$d_{1,3}$	$d_{1,4}$	$d_{1,5}$
$d_{2,1}$	$d_{2,2}$	$d_{2,3}$	$d_{2,4}$	$d_{2,5}$
$d_{3,1}$	$d_{3,2}$	$d_{3,3}$	$d_{3,4}$	$d_{3,5}$
$d_{4,1}$	$d_{4,2}$	$d_{4,3}$	$d_{4,4}$	$d_{4,5}$
$d_{5,1}$	$d_{5,2}$	$d_{5,3}$	$d_{5,4}$	$d_{5,5}$

=

$\mathbf{A}\mathbf{C}_1\mathbf{B}$

$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,3}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,4}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,5}$
$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,2}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,3}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,4}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,5}$
$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,1}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,2}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,3}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,4}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,5}$
$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,1}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,2}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,3}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,4}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,5}$
$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,1}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,2}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,3}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,4}$	$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,5}$

Algebraic cryptanalysis : modelisation example

Example. $\mathbf{D}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}$
$\mathbf{D}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}$

$$d_{1,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} = 0, \qquad d_{1,2} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2} = 0, \ldots$$

$$d_{2,1} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} = 0, \qquad d_{l,p} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{l,j}c_{j,i}b_{i,p} = 0$$

$$\underbrace{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} \end{bmatrix}}_{\mathbf{D}_1} = \underbrace{\begin{bmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{1,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{2,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{3,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{4,j}c_{j,i}b_{i,5} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,1} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,3} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,4} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{5,j}c_{j,i}b_{i,5} \end{bmatrix}}_{\mathbf{A}\mathbf{C}_1\mathbf{B}}$$

Algebraic cryptanalysis : modelisation example

A motivating example: a better idea for modelisation.

Given matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2 \in \mathscr{M}_{n,n}(\mathbb{F}_q)$ (the space of matrices over $\mathbb{F}_q$ of size $n \times n$), find $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ (the space of invertible matrices over $\mathbb{F}_q$ of size $n \times n$), such that

$$\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$$

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix}}_{\mathbf{A}^{-1}} \underbrace{\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} \end{bmatrix}}_{\mathbf{D}_1} = \underbrace{\begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & c_{3,5} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & c_{4,5} \\ c_{5,1} & c_{5,2} & c_{5,3} & c_{5,4} & c_{5,5} \end{bmatrix}}_{\mathbf{C}_1} \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & b_{5,5} \end{bmatrix}}_{\mathbf{B}}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$$

$$\overbrace{\begin{bmatrix}
\sum_{i=1}^{n} a_{1,i}d_{i,1} & \sum_{i=1}^{n} a_{1,i}d_{i,2} & \sum_{i=1}^{n} a_{1,i}d_{i,3} & \sum_{i=1}^{n} a_{1,i}d_{i,4} & \sum_{i=1}^{n} a_{1,i}d_{i,5} \\
\sum_{i=1}^{n} a_{2,i}d_{i,1} & \sum_{i=1}^{n} a_{2,i}d_{i,2} & \sum_{i=1}^{n} a_{2,i}d_{i,3} & \sum_{i=1}^{n} a_{2,i}d_{i,4} & \sum_{i=1}^{n} a_{2,i}d_{i,5} \\
\sum_{i=1}^{n} a_{3,i}d_{i,1} & \sum_{i=1}^{n} a_{3,i}d_{i,2} & \sum_{i=1}^{n} a_{3,i}d_{i,3} & \sum_{i=1}^{n} a_{3,i}d_{i,4} & \sum_{i=1}^{n} a_{3,i}d_{i,5} \\
\sum_{i=1}^{n} a_{4,i}d_{i,1} & \sum_{i=1}^{n} a_{4,i}d_{i,2} & \sum_{i=1}^{n} a_{4,i}d_{i,3} & \sum_{i=1}^{n} a_{4,i}d_{i,4} & \sum_{i=1}^{n} a_{4,i}d_{i,5} \\
\sum_{i=1}^{n} a_{5,i}d_{i,1} & \sum_{i=1}^{n} a_{5,i}d_{i,2} & \sum_{i=1}^{n} a_{5,i}d_{i,3} & \sum_{i=1}^{n} a_{5,i}d_{i,4} & \sum_{i=1}^{n} a_{5,i}d_{i,5}
\end{bmatrix}}^{\mathbf{A}^{-1}\mathbf{D}_1}
=
\overbrace{\begin{bmatrix}
\sum_{i=1}^{n} c_{1,i}b_{i,1} & \sum_{i=1}^{n} c_{1,i}b_{i,2} & \sum_{i=1}^{n} c_{1,i}b_{i,3} & \sum_{i=1}^{n} c_{1,i}b_{i,4} & \sum_{i=1}^{n} c_{1,i}b_{i,5} \\
\sum_{i=1}^{n} c_{2,i}b_{i,1} & \sum_{i=1}^{n} c_{2,i}b_{i,2} & \sum_{i=1}^{n} c_{2,i}b_{i,3} & \sum_{i=1}^{n} c_{2,i}b_{i,4} & \sum_{i=1}^{n} c_{2,i}b_{i,5} \\
\sum_{i=1}^{n} c_{3,i}b_{i,1} & \sum_{i=1}^{n} c_{3,i}b_{i,2} & \sum_{i=1}^{n} c_{3,i}b_{i,3} & \sum_{i=1}^{n} c_{3,i}b_{i,4} & \sum_{i=1}^{n} c_{3,i}b_{i,5} \\
\sum_{i=1}^{n} c_{4,i}b_{i,1} & \sum_{i=1}^{n} c_{4,i}b_{i,2} & \sum_{i=1}^{n} c_{4,i}b_{i,3} & \sum_{i=1}^{n} c_{4,i}b_{i,4} & \sum_{i=1}^{n} c_{4,i}b_{i,5} \\
\sum_{i=1}^{n} c_{5,i}b_{i,1} & \sum_{i=1}^{n} c_{5,i}b_{i,2} & \sum_{i=1}^{n} c_{5,i}b_{i,3} & \sum_{i=1}^{n} c_{5,i}b_{i,4} & \sum_{i=1}^{n} c_{5,i}b_{i,5}
\end{bmatrix}}^{\mathbf{C}_1\mathbf{B}}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$$

$$\sum_{i=1}^{n} a_{1,i}d_{i,1} - \sum_{i=1}^{n} c_{1,i}b_{i,1} = 0,$$

$$\mathbf{A}^{-1}\mathbf{D}_1 \qquad\qquad \mathbf{C}_1\,\mathbf{B}$$

$$\begin{bmatrix}
\sum_{i=1}^{n} a_{1,i}d_{i,1} & \sum_{i=1}^{n} a_{1,i}d_{i,2} & \sum_{i=1}^{n} a_{1,i}d_{i,3} & \sum_{i=1}^{n} a_{1,i}d_{i,4} & \sum_{i=1}^{n} a_{1,i}d_{i,5} \\
\sum_{i=1}^{n} a_{2,i}d_{i,1} & \sum_{i=1}^{n} a_{2,i}d_{i,2} & \sum_{i=1}^{n} a_{2,i}d_{i,3} & \sum_{i=1}^{n} a_{2,i}d_{i,4} & \sum_{i=1}^{n} a_{2,i}d_{i,5} \\
\sum_{i=1}^{n} a_{3,i}d_{i,1} & \sum_{i=1}^{n} a_{3,i}d_{i,2} & \sum_{i=1}^{n} a_{3,i}d_{i,3} & \sum_{i=1}^{n} a_{3,i}d_{i,4} & \sum_{i=1}^{n} a_{3,i}d_{i,5} \\
\sum_{i=1}^{n} a_{4,i}d_{i,1} & \sum_{i=1}^{n} a_{4,i}d_{i,2} & \sum_{i=1}^{n} a_{4,i}d_{i,3} & \sum_{i=1}^{n} a_{4,i}d_{i,4} & \sum_{i=1}^{n} a_{4,i}d_{i,5} \\
\sum_{i=1}^{n} a_{5,i}d_{i,1} & \sum_{i=1}^{n} a_{5,i}d_{i,2} & \sum_{i=1}^{n} a_{5,i}d_{i,3} & \sum_{i=1}^{n} a_{5,i}d_{i,4} & \sum_{i=1}^{n} a_{5,i}d_{i,5}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{i=1}^{n} c_{1,i}b_{i,1} & \sum_{i=1}^{n} c_{1,i}b_{i,2} & \sum_{i=1}^{n} c_{1,i}b_{i,3} & \sum_{i=1}^{n} c_{1,i}b_{i,4} & \sum_{i=1}^{n} c_{1,i}b_{i,5} \\
\sum_{i=1}^{n} c_{2,i}b_{i,1} & \sum_{i=1}^{n} c_{2,i}b_{i,2} & \sum_{i=1}^{n} c_{2,i}b_{i,3} & \sum_{i=1}^{n} c_{2,i}b_{i,4} & \sum_{i=1}^{n} c_{2,i}b_{i,5} \\
\sum_{i=1}^{n} c_{3,i}b_{i,1} & \sum_{i=1}^{n} c_{3,i}b_{i,2} & \sum_{i=1}^{n} c_{3,i}b_{i,3} & \sum_{i=1}^{n} c_{3,i}b_{i,4} & \sum_{i=1}^{n} c_{3,i}b_{i,5} \\
\sum_{i=1}^{n} c_{4,i}b_{i,1} & \sum_{i=1}^{n} c_{4,i}b_{i,2} & \sum_{i=1}^{n} c_{4,i}b_{i,3} & \sum_{i=1}^{n} c_{4,i}b_{i,4} & \sum_{i=1}^{n} c_{4,i}b_{i,5} \\
\sum_{i=1}^{n} c_{5,i}b_{i,1} & \sum_{i=1}^{n} c_{5,i}b_{i,2} & \sum_{i=1}^{n} c_{5,i}b_{i,3} & \sum_{i=1}^{n} c_{5,i}b_{i,4} & \sum_{i=1}^{n} c_{5,i}b_{i,5}
\end{bmatrix}$$

Algebraic cryptanalysis : modelisation example

Example. $\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$

$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$

$$\sum_{i=1}^{n} a_{1,i}d_{i,1} - \sum_{i=1}^{n} c_{1,i}b_{i,1} = 0,$$

$$\sum_{i=1}^{n} a_{2,i}d_{i,1} - \sum_{i=1}^{n} c_{2,i}b_{i,1} = 0,$$

$$\overset{\mathbf{A}^{-1}\mathbf{D}_1}{\begin{bmatrix}
\sum_{i=1}^{n} a_{1,i}d_{i,1} & \sum_{i=1}^{n} a_{1,i}d_{i,2} & \sum_{i=1}^{n} a_{1,i}d_{i,3} & \sum_{i=1}^{n} a_{1,i}d_{i,4} & \sum_{i=1}^{n} a_{1,i}d_{i,5} \\
\sum_{i=1}^{n} a_{2,i}d_{i,1} & \sum_{i=1}^{n} a_{2,i}d_{i,2} & \sum_{i=1}^{n} a_{2,i}d_{i,3} & \sum_{i=1}^{n} a_{2,i}d_{i,4} & \sum_{i=1}^{n} a_{2,i}d_{i,5} \\
\sum_{i=1}^{n} a_{3,i}d_{i,1} & \sum_{i=1}^{n} a_{3,i}d_{i,2} & \sum_{i=1}^{n} a_{3,i}d_{i,3} & \sum_{i=1}^{n} a_{3,i}d_{i,4} & \sum_{i=1}^{n} a_{3,i}d_{i,5} \\
\sum_{i=1}^{n} a_{4,i}d_{i,1} & \sum_{i=1}^{n} a_{4,i}d_{i,2} & \sum_{i=1}^{n} a_{4,i}d_{i,3} & \sum_{i=1}^{n} a_{4,i}d_{i,4} & \sum_{i=1}^{n} a_{4,i}d_{i,5} \\
\sum_{i=1}^{n} a_{5,i}d_{i,1} & \sum_{i=1}^{n} a_{5,i}d_{i,2} & \sum_{i=1}^{n} a_{5,i}d_{i,3} & \sum_{i=1}^{n} a_{5,i}d_{i,4} & \sum_{i=1}^{n} a_{5,i}d_{i,5}
\end{bmatrix}}
=
\overset{\mathbf{C}_1\mathbf{B}}{\begin{bmatrix}
\sum_{i=1}^{n} c_{1,i}b_{i,1} & \sum_{i=1}^{n} c_{1,i}b_{i,2} & \sum_{i=1}^{n} c_{1,i}b_{i,3} & \sum_{i=1}^{n} c_{1,i}b_{i,4} & \sum_{i=1}^{n} c_{1,i}b_{i,5} \\
\sum_{i=1}^{n} c_{2,i}b_{i,1} & \sum_{i=1}^{n} c_{2,i}b_{i,2} & \sum_{i=1}^{n} c_{2,i}b_{i,3} & \sum_{i=1}^{n} c_{2,i}b_{i,4} & \sum_{i=1}^{n} c_{2,i}b_{i,5} \\
\sum_{i=1}^{n} c_{3,i}b_{i,1} & \sum_{i=1}^{n} c_{3,i}b_{i,2} & \sum_{i=1}^{n} c_{3,i}b_{i,3} & \sum_{i=1}^{n} c_{3,i}b_{i,4} & \sum_{i=1}^{n} c_{3,i}b_{i,5} \\
\sum_{i=1}^{n} c_{4,i}b_{i,1} & \sum_{i=1}^{n} c_{4,i}b_{i,2} & \sum_{i=1}^{n} c_{4,i}b_{i,3} & \sum_{i=1}^{n} c_{4,i}b_{i,4} & \sum_{i=1}^{n} c_{4,i}b_{i,5} \\
\sum_{i=1}^{n} c_{5,i}b_{i,1} & \sum_{i=1}^{n} c_{5,i}b_{i,2} & \sum_{i=1}^{n} c_{5,i}b_{i,3} & \sum_{i=1}^{n} c_{5,i}b_{i,4} & \sum_{i=1}^{n} c_{5,i}b_{i,5}
\end{bmatrix}}$$

Algebraic cryptanalysis : modelisation example

Example.
$$\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$$

$$\sum_{i=1}^{n} a_{1,i}d_{i,1} - \sum_{i=1}^{n} c_{1,i}b_{i,1} = 0, \quad \sum_{i=1}^{n} a_{1,i}d_{i,2} - \sum_{i=1}^{n} c_{1,i}b_{i,2} = 0, \dots$$
$$\sum_{i=1}^{n} a_{2,i}d_{i,1} - \sum_{i=1}^{n} c_{2,i}b_{i,1} = 0,$$

$$\mathbf{A}^{-1}\mathbf{D}_1 = \begin{bmatrix} \sum_{i=1}^{n} a_{1,i}d_{i,1} & \sum_{i=1}^{n} a_{1,i}d_{i,2} & \sum_{i=1}^{n} a_{1,i}d_{i,3} & \sum_{i=1}^{n} a_{1,i}d_{i,4} & \sum_{i=1}^{n} a_{1,i}d_{i,5} \\ \sum_{i=1}^{n} a_{2,i}d_{i,1} & \sum_{i=1}^{n} a_{2,i}d_{i,2} & \sum_{i=1}^{n} a_{2,i}d_{i,3} & \sum_{i=1}^{n} a_{2,i}d_{i,4} & \sum_{i=1}^{n} a_{2,i}d_{i,5} \\ \sum_{i=1}^{n} a_{3,i}d_{i,1} & \sum_{i=1}^{n} a_{3,i}d_{i,2} & \sum_{i=1}^{n} a_{3,i}d_{i,3} & \sum_{i=1}^{n} a_{3,i}d_{i,4} & \sum_{i=1}^{n} a_{3,i}d_{i,5} \\ \sum_{i=1}^{n} a_{4,i}d_{i,1} & \sum_{i=1}^{n} a_{4,i}d_{i,2} & \sum_{i=1}^{n} a_{4,i}d_{i,3} & \sum_{i=1}^{n} a_{4,i}d_{i,4} & \sum_{i=1}^{n} a_{4,i}d_{i,5} \\ \sum_{i=1}^{n} a_{5,i}d_{i,1} & \sum_{i=1}^{n} a_{5,i}d_{i,2} & \sum_{i=1}^{n} a_{5,i}d_{i,3} & \sum_{i=1}^{n} a_{5,i}d_{i,4} & \sum_{i=1}^{n} a_{5,i}d_{i,5} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} c_{1,i}b_{i,1} & \sum_{i=1}^{n} c_{1,i}b_{i,2} & \sum_{i=1}^{n} c_{1,i}b_{i,3} & \sum_{i=1}^{n} c_{1,i}b_{i,4} & \sum_{i=1}^{n} c_{1,i}b_{i,5} \\ \sum_{i=1}^{n} c_{2,i}b_{i,1} & \sum_{i=1}^{n} c_{2,i}b_{i,2} & \sum_{i=1}^{n} c_{2,i}b_{i,3} & \sum_{i=1}^{n} c_{2,i}b_{i,4} & \sum_{i=1}^{n} c_{2,i}b_{i,5} \\ \sum_{i=1}^{n} c_{3,i}b_{i,1} & \sum_{i=1}^{n} c_{3,i}b_{i,2} & \sum_{i=1}^{n} c_{3,i}b_{i,3} & \sum_{i=1}^{n} c_{3,i}b_{i,4} & \sum_{i=1}^{n} c_{3,i}b_{i,5} \\ \sum_{i=1}^{n} c_{4,i}b_{i,1} & \sum_{i=1}^{n} c_{4,i}b_{i,2} & \sum_{i=1}^{n} c_{4,i}b_{i,3} & \sum_{i=1}^{n} c_{4,i}b_{i,4} & \sum_{i=1}^{n} c_{4,i}b_{i,5} \\ \sum_{i=1}^{n} c_{5,i}b_{i,1} & \sum_{i=1}^{n} c_{5,i}b_{i,2} & \sum_{i=1}^{n} c_{5,i}b_{i,3} & \sum_{i=1}^{n} c_{5,i}b_{i,4} & \sum_{i=1}^{n} c_{5,i}b_{i,5} \end{bmatrix} = \mathbf{C}_1\mathbf{B}$$

Algebraic cryptanalysis : modelisation example

Example. $\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$

$\mathbf{A}^{-1}\mathbf{D}_2 = \mathbf{C}_2\mathbf{B}$

$$\sum_{i=1}^{n} a_{1,i}d_{i,1} - \sum_{i=1}^{n} c_{1,i}b_{i,1} = 0, \quad \sum_{i=1}^{n} a_{1,i}d_{i,2} - \sum_{i=1}^{n} c_{1,i}b_{i,2} = 0, \ldots$$

$$\sum_{i=1}^{n} a_{2,i}d_{i,1} - \sum_{i=1}^{n} c_{2,i}b_{i,1} = 0, \quad \sum_{i=1}^{n} a_{l,i}d_{i,p} - \sum_{i=1}^{n} c_{l,i}b_{i,p} = 0$$

$$\underbrace{\begin{bmatrix} \sum_{i=1}^{n} a_{1,i}d_{i,1} & \sum_{i=1}^{n} a_{1,i}d_{i,2} & \sum_{i=1}^{n} a_{1,i}d_{i,3} & \sum_{i=1}^{n} a_{1,i}d_{i,4} & \sum_{i=1}^{n} a_{1,i}d_{i,5} \\ \sum_{i=1}^{n} a_{2,i}d_{i,1} & \sum_{i=1}^{n} a_{2,i}d_{i,2} & \sum_{i=1}^{n} a_{2,i}d_{i,3} & \sum_{i=1}^{n} a_{2,i}d_{i,4} & \sum_{i=1}^{n} a_{2,i}d_{i,5} \\ \sum_{i=1}^{n} a_{3,i}d_{i,1} & \sum_{i=1}^{n} a_{3,i}d_{i,2} & \sum_{i=1}^{n} a_{3,i}d_{i,3} & \sum_{i=1}^{n} a_{3,i}d_{i,4} & \sum_{i=1}^{n} a_{3,i}d_{i,5} \\ \sum_{i=1}^{n} a_{4,i}d_{i,1} & \sum_{i=1}^{n} a_{4,i}d_{i,2} & \sum_{i=1}^{n} a_{4,i}d_{i,3} & \sum_{i=1}^{n} a_{4,i}d_{i,4} & \sum_{i=1}^{n} a_{4,i}d_{i,5} \\ \sum_{i=1}^{n} a_{5,i}d_{i,1} & \sum_{i=1}^{n} a_{5,i}d_{i,2} & \sum_{i=1}^{n} a_{5,i}d_{i,3} & \sum_{i=1}^{n} a_{5,i}d_{i,4} & \sum_{i=1}^{n} a_{5,i}d_{i,5} \end{bmatrix}}_{\mathbf{A}^{-1}\mathbf{D}_1} = \underbrace{\begin{bmatrix} \sum_{i=1}^{n} c_{1,i}b_{i,1} & \sum_{i=1}^{n} c_{1,i}b_{i,2} & \sum_{i=1}^{n} c_{1,i}b_{i,3} & \sum_{i=1}^{n} c_{1,i}b_{i,4} & \sum_{i=1}^{n} c_{1,i}b_{i,5} \\ \sum_{i=1}^{n} c_{2,i}b_{i,1} & \sum_{i=1}^{n} c_{2,i}b_{i,2} & \sum_{i=1}^{n} c_{2,i}b_{i,3} & \sum_{i=1}^{n} c_{2,i}b_{i,4} & \sum_{i=1}^{n} c_{2,i}b_{i,5} \\ \sum_{i=1}^{n} c_{3,i}b_{i,1} & \sum_{i=1}^{n} c_{3,i}b_{i,2} & \sum_{i=1}^{n} c_{3,i}b_{i,3} & \sum_{i=1}^{n} c_{3,i}b_{i,4} & \sum_{i=1}^{n} c_{3,i}b_{i,5} \\ \sum_{i=1}^{n} c_{4,i}b_{i,1} & \sum_{i=1}^{n} c_{4,i}b_{i,2} & \sum_{i=1}^{n} c_{4,i}b_{i,3} & \sum_{i=1}^{n} c_{4,i}b_{i,4} & \sum_{i=1}^{n} c_{4,i}b_{i,5} \\ \sum_{i=1}^{n} c_{5,i}b_{i,1} & \sum_{i=1}^{n} c_{5,i}b_{i,2} & \sum_{i=1}^{n} c_{5,i}b_{i,3} & \sum_{i=1}^{n} c_{5,i}b_{i,4} & \sum_{i=1}^{n} c_{5,i}b_{i,5} \end{bmatrix}}_{\mathbf{C}_1\,\mathbf{B}}$$

Algebraic cryptanalysis : modelisation example

A motivating example: a better idea for modelisation.

Given matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2 \in \mathscr{M}_{n,n}(\mathbb{F}_q)$ (the space of matrices over $\mathbb{F}_q$ of size $n \times n$), find $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ (the space of invertible matrices over $\mathbb{F}_q$ of size $n \times n$), such that

$$\mathbf{A}^{-1}\mathbf{D}_1 = \mathbf{C}_1\mathbf{B}$$
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→ Results in a linear system with the same number of variables and equations.

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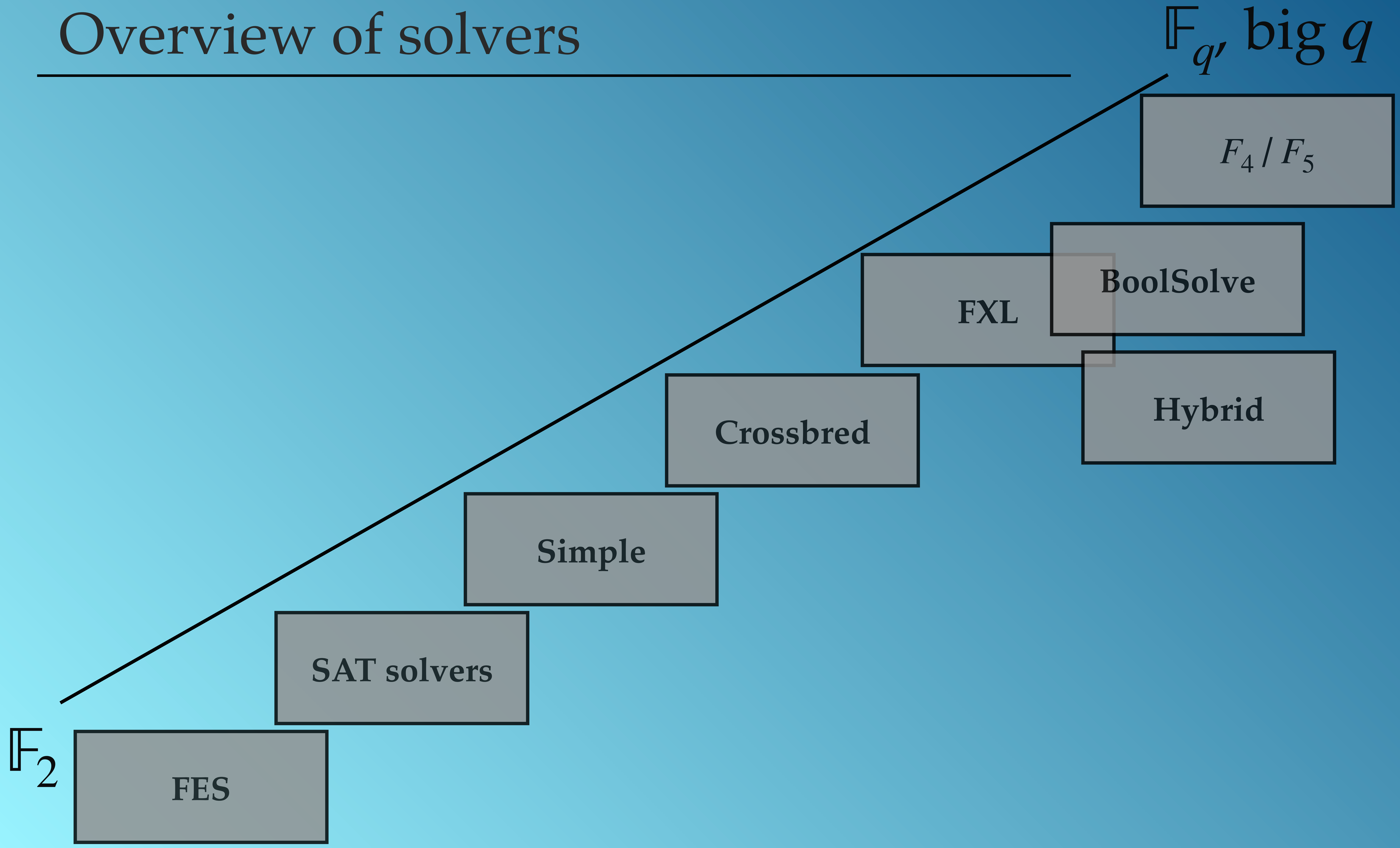
→ Results in a linear system with the same number of variables and equations.

- → If $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2$ are all full rank, we should have a unique solution.
- → We can easily recover $\mathbf{A}$ from $\mathbf{A}^{-1}$.

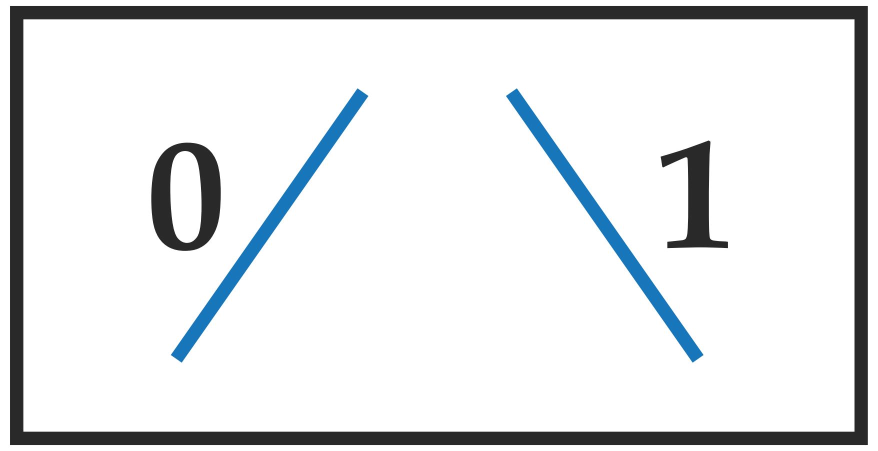
Overview of solvers

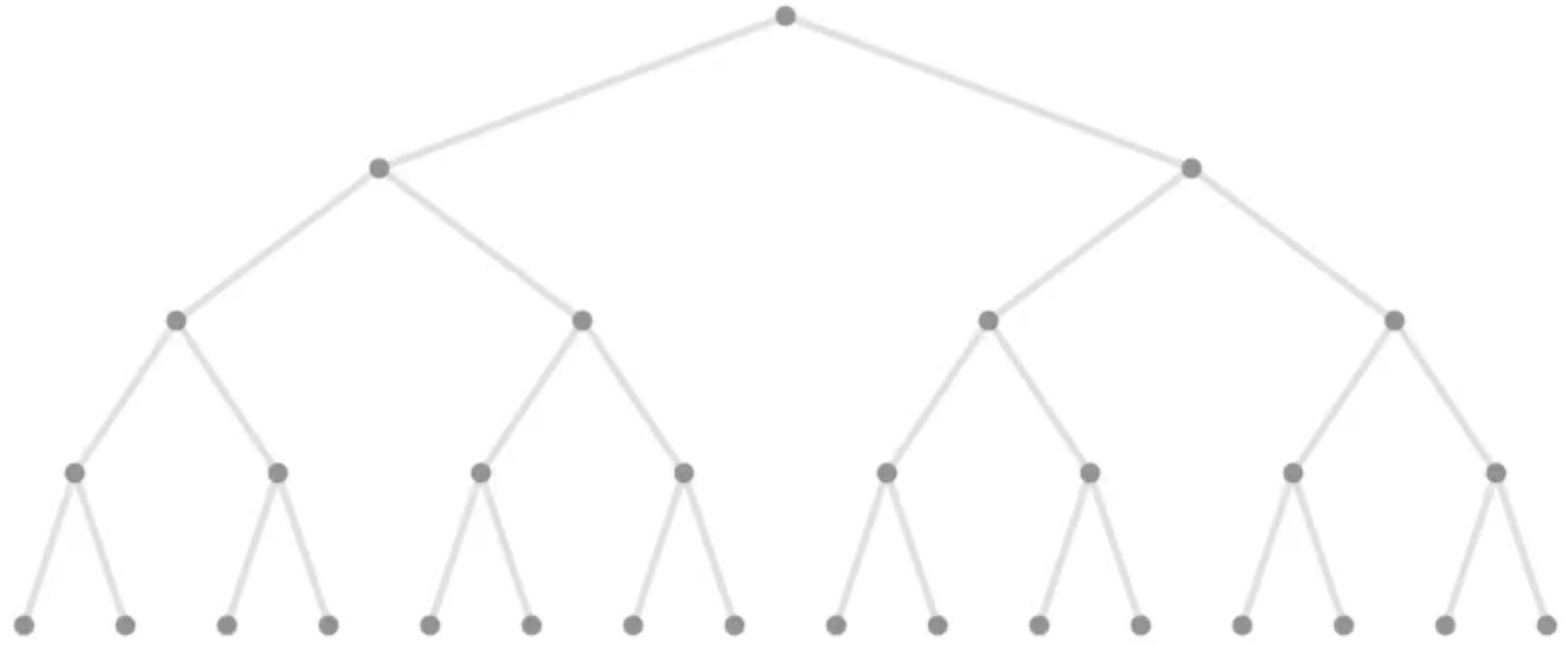


(Fast) Exhaustive Search

[Bouillaguet, Chen, Cheng, Chou, Niederhagen, Shamir, Yang, 2010]

Exhaustive Search

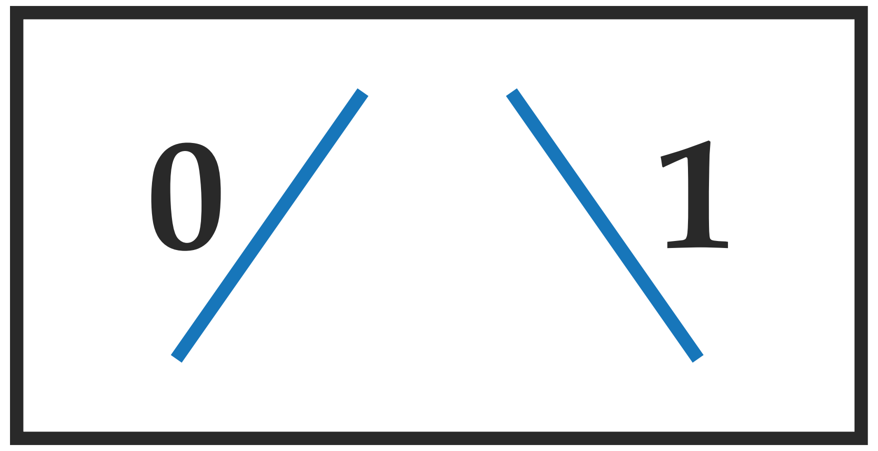




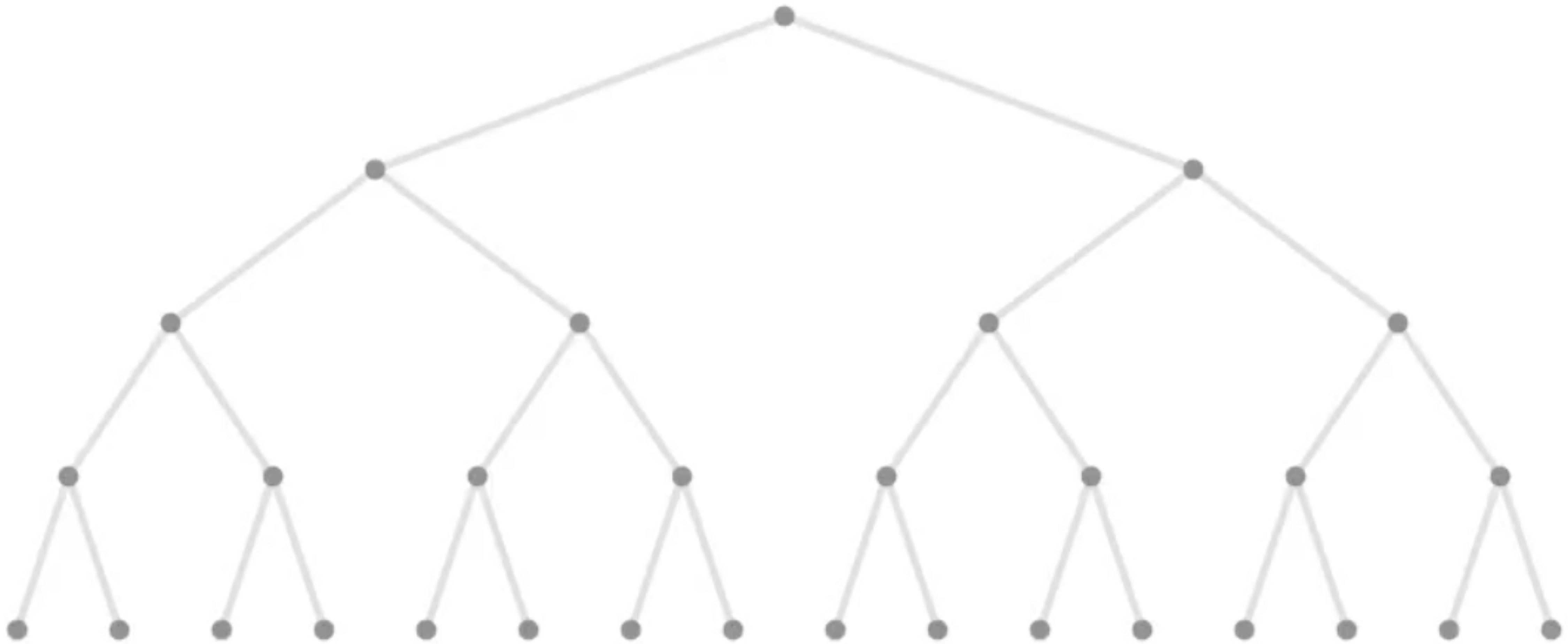
$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$
$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$
$$x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_4 = 0$$
$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

Binary search tree

Exhaustive Search



Worst-case complexity: $\mathcal{O}(2^n)$



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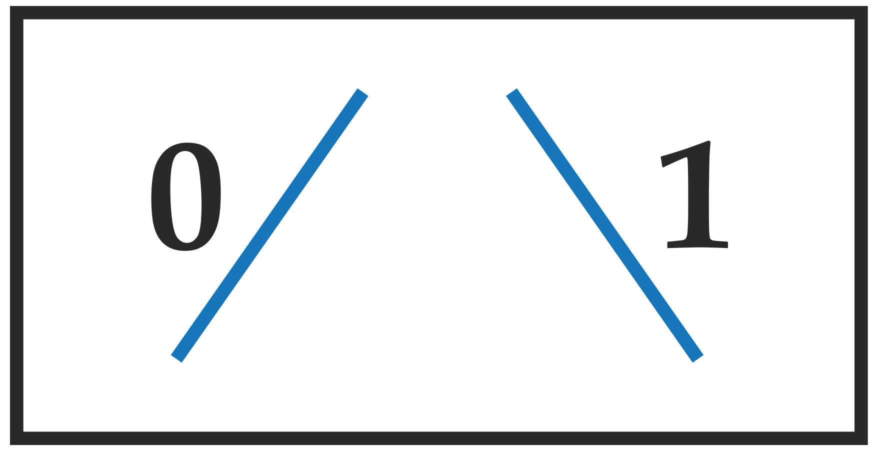

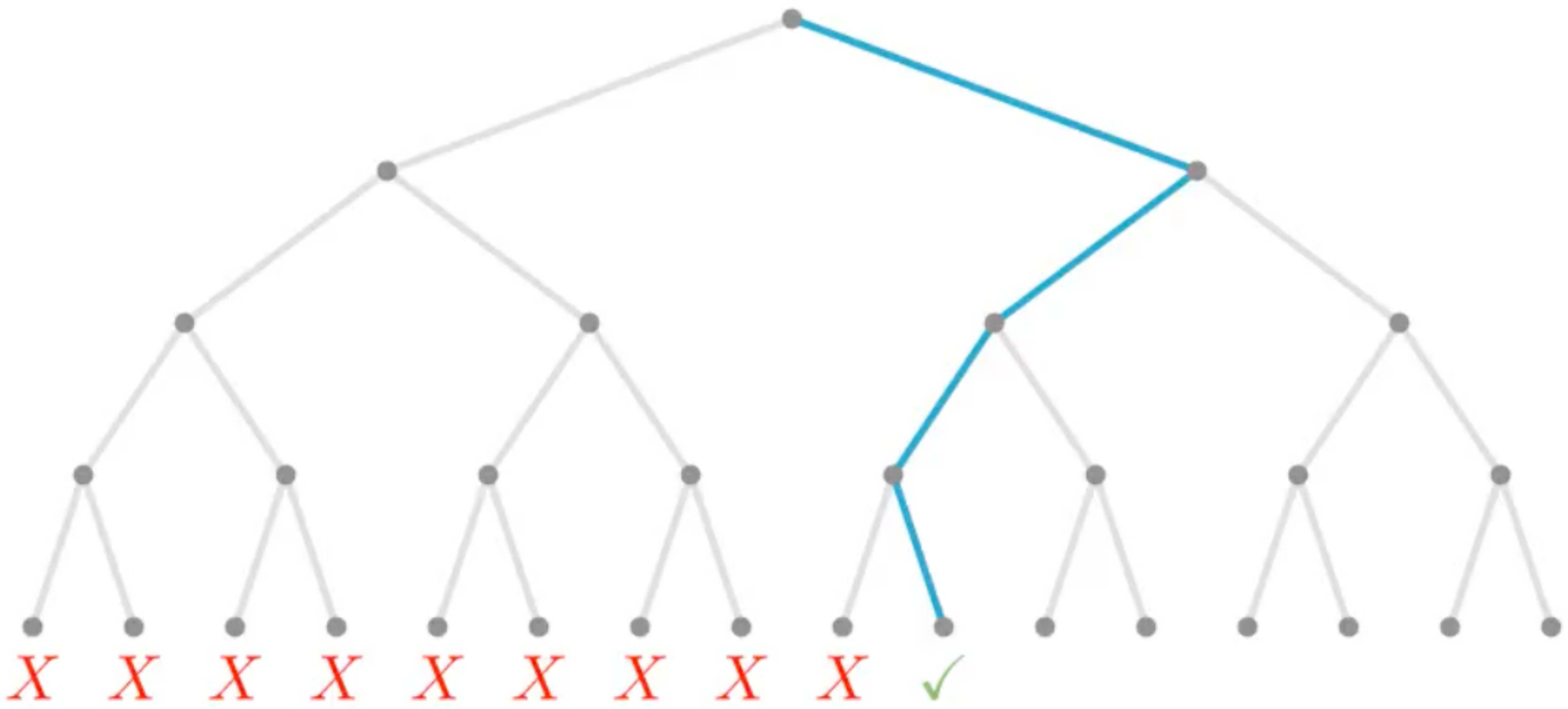

Binary search tree

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

$$0 \cdot 0 + 0 \cdot 1 + 1 + 0 + 1 = 0$$

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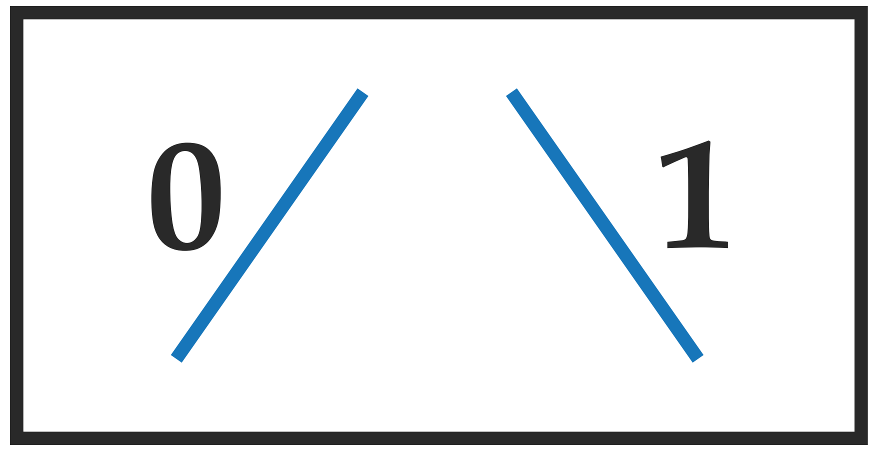

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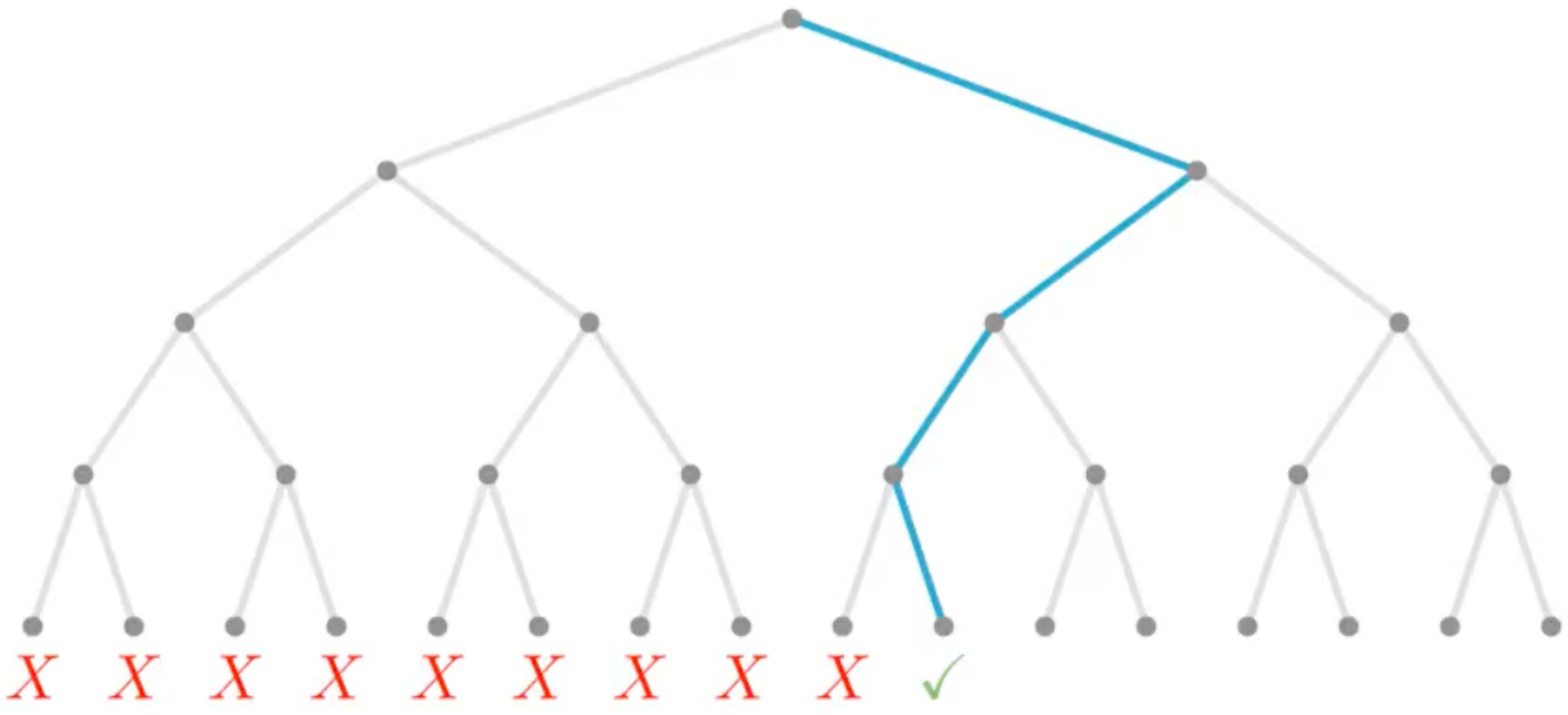

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Fast Exhaustive Search

* The libFES solver

Gray code

- An ordering of the binary system where two successive values differ in only one bit.

Example. $n = 4$

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000

Fast Exhaustive Search

Gray code

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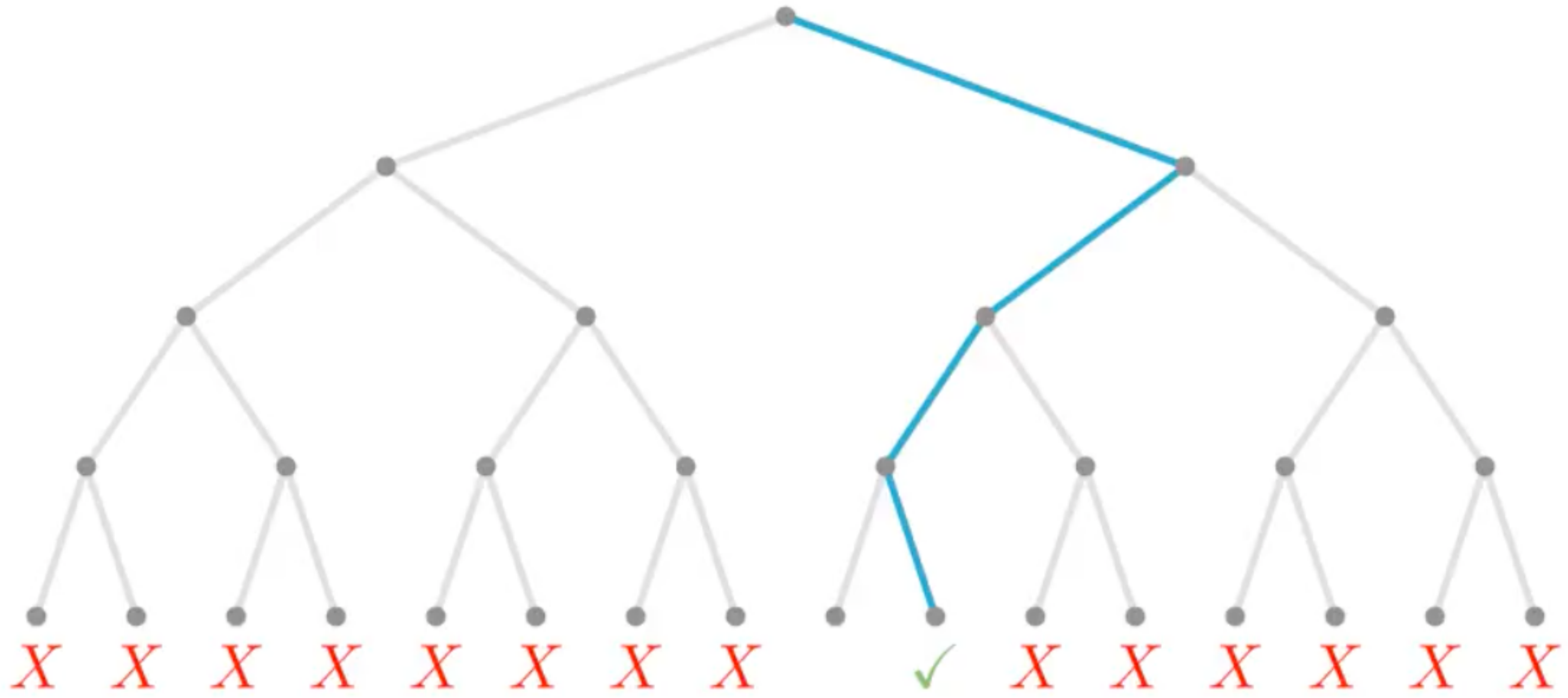

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Fast Exhaustive Search

Worst-case complexity: $\mathcal{O}(2^n)$

! But, it differs from the depth-first traversal in the polynomial factors

Gray code

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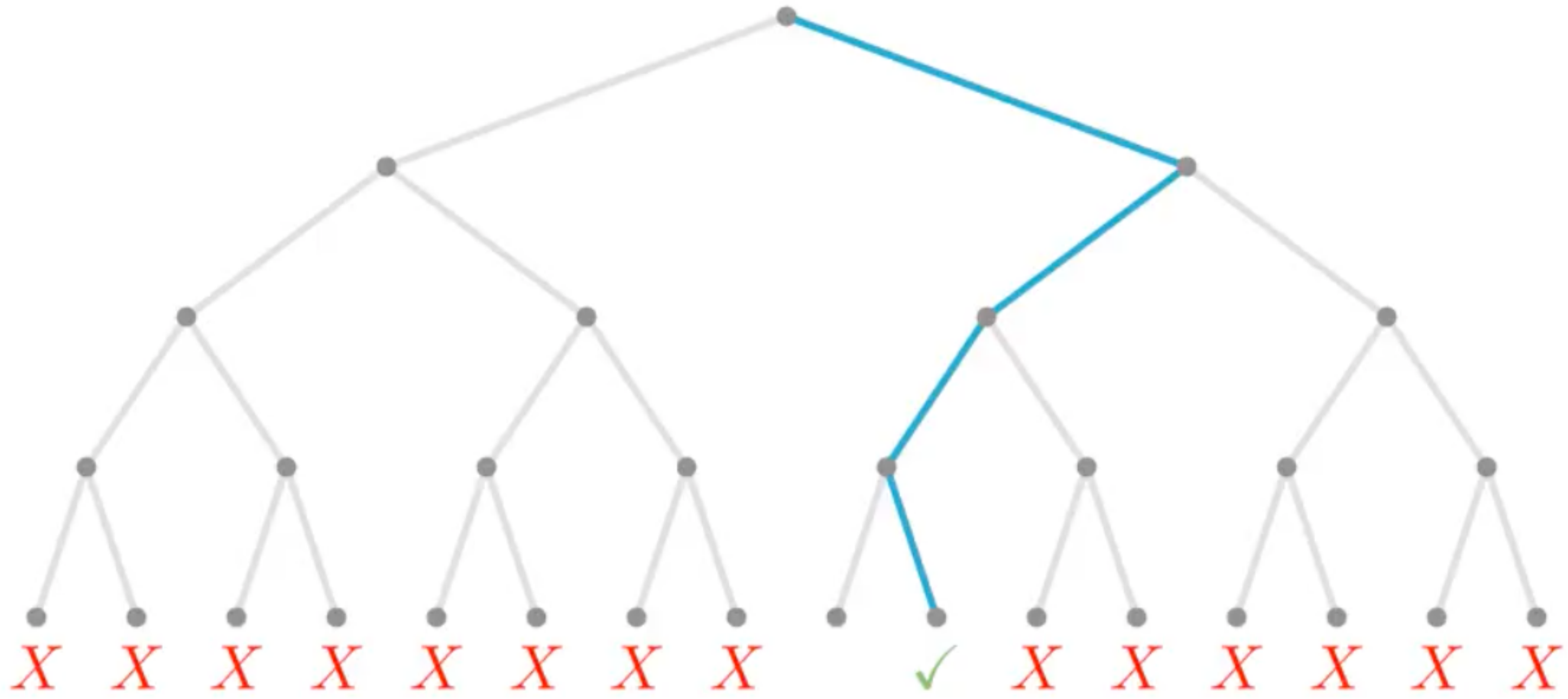

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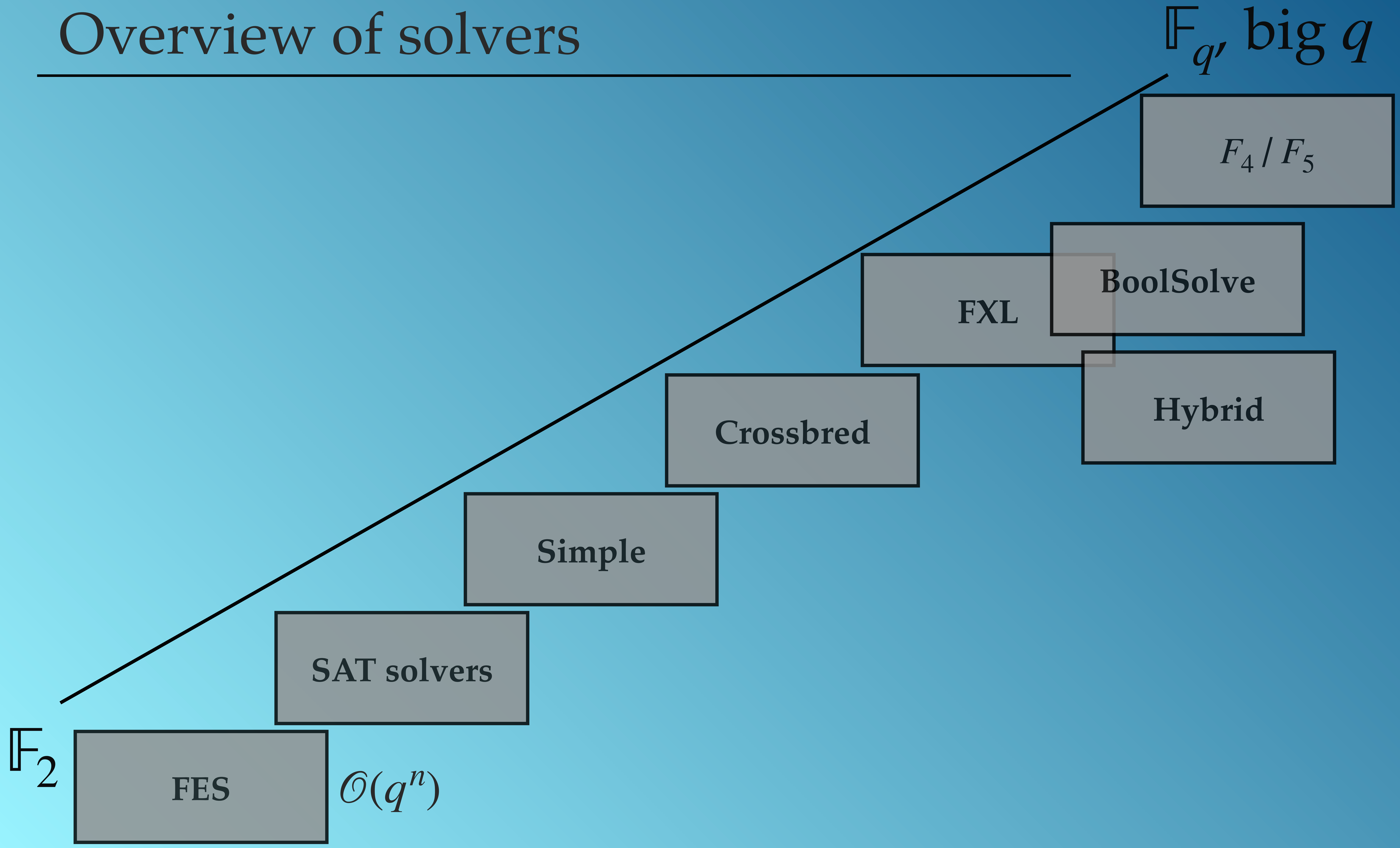
Overview of solvers



SAT solvers

CryptoMiniSat [Soos, Nohl, Castelluccia, 2009], WDSat [T., Dequen, Ionica, 2020]

Simple algorithm

[Bouillaguet, Delaplace, T., 2021]

(SAT solvers)

- Propositional formula in Conjunctive Normal Form (CNF): a conjunction of clauses where each clause is a disjunction of literals and where each literal is a variable or a negated variable.

Example.
$$
\begin{aligned}
&(x_1 \vee \neg x_2) \wedge \\
&(x_2 \vee x_3 \vee x_4) \wedge \\
&(\neg x_1 \vee x_4)
\end{aligned}
$$

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The SATisfiability problem

Given a propositional formula, determine whether there exists an interpretation (assignment of all variables) such that the formula is satisfied (evaluates to TRUE).

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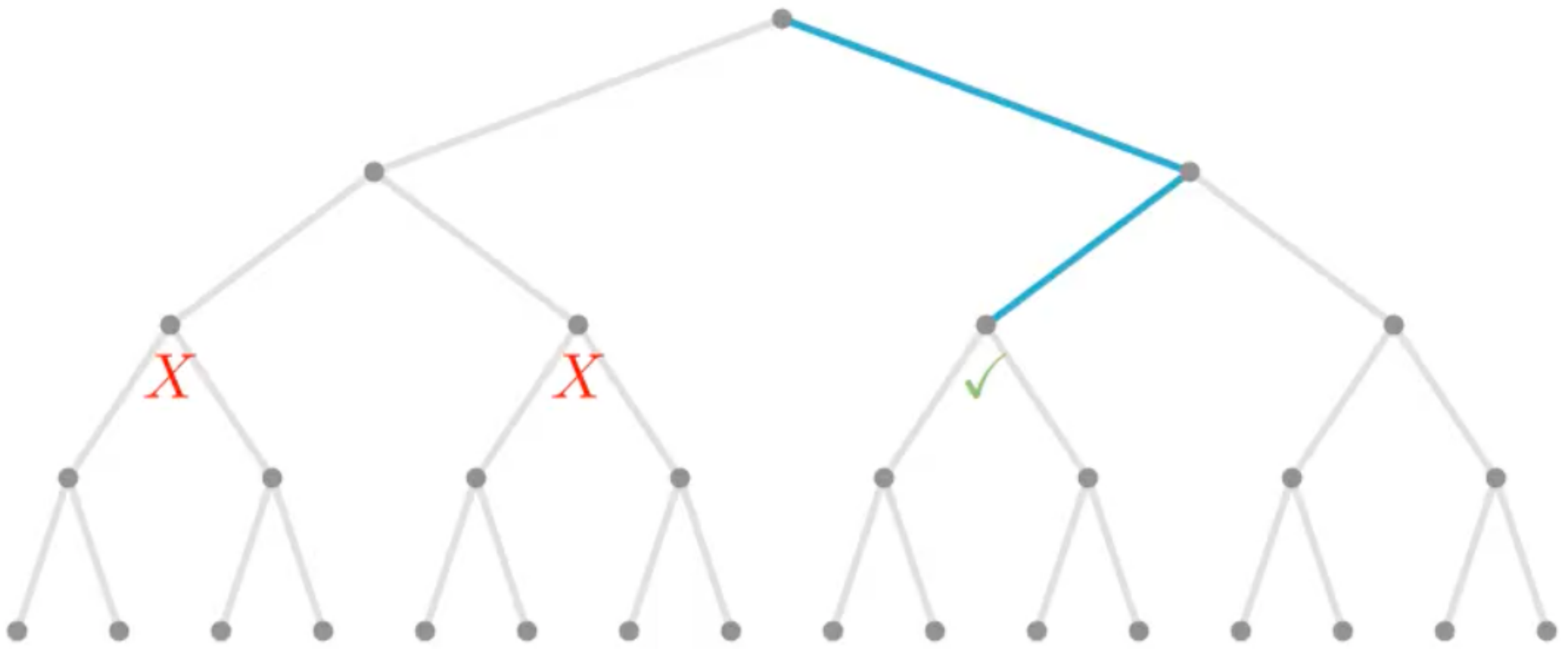
SAT solver: a tool for solving the SAT problem.

Partial assignment and conflicts



$$1 \cdot 0 + 1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$
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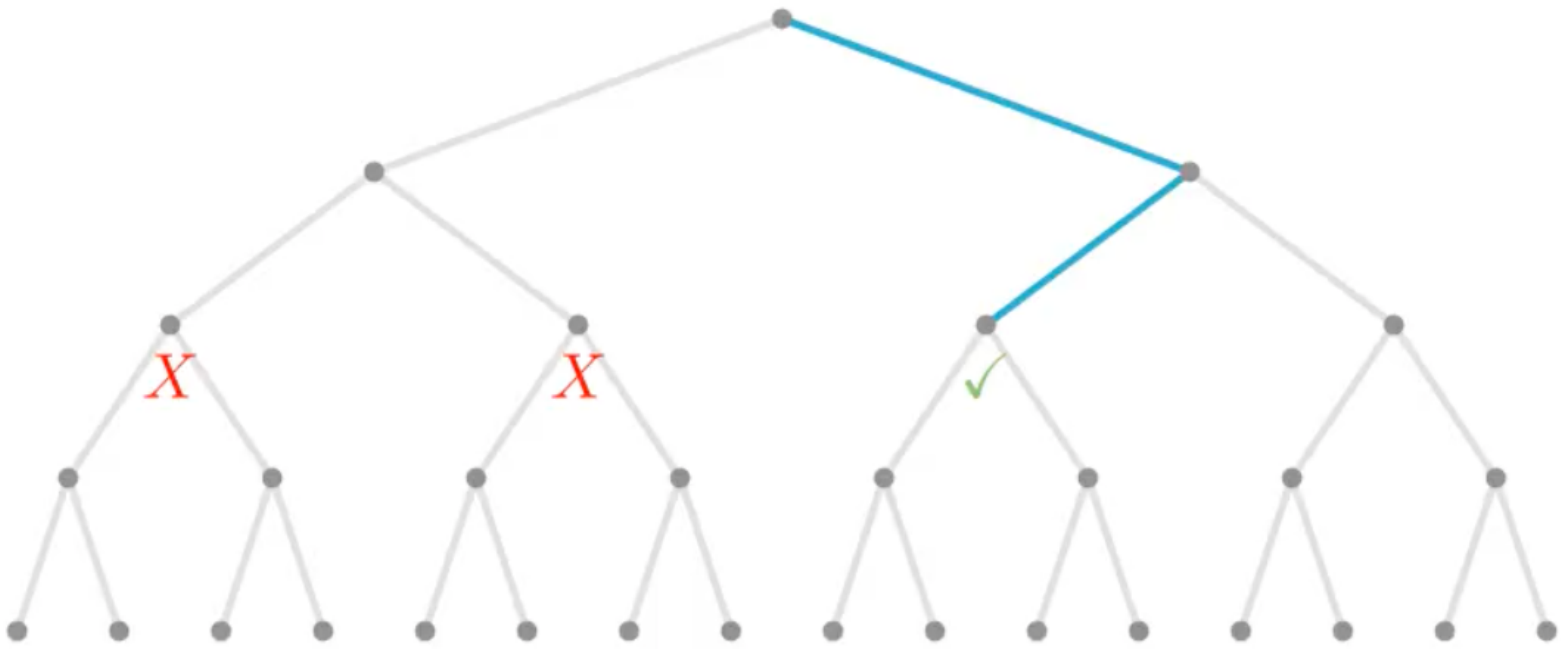
Partial assignment and conflicts

Which (portion of) branches are missing ??



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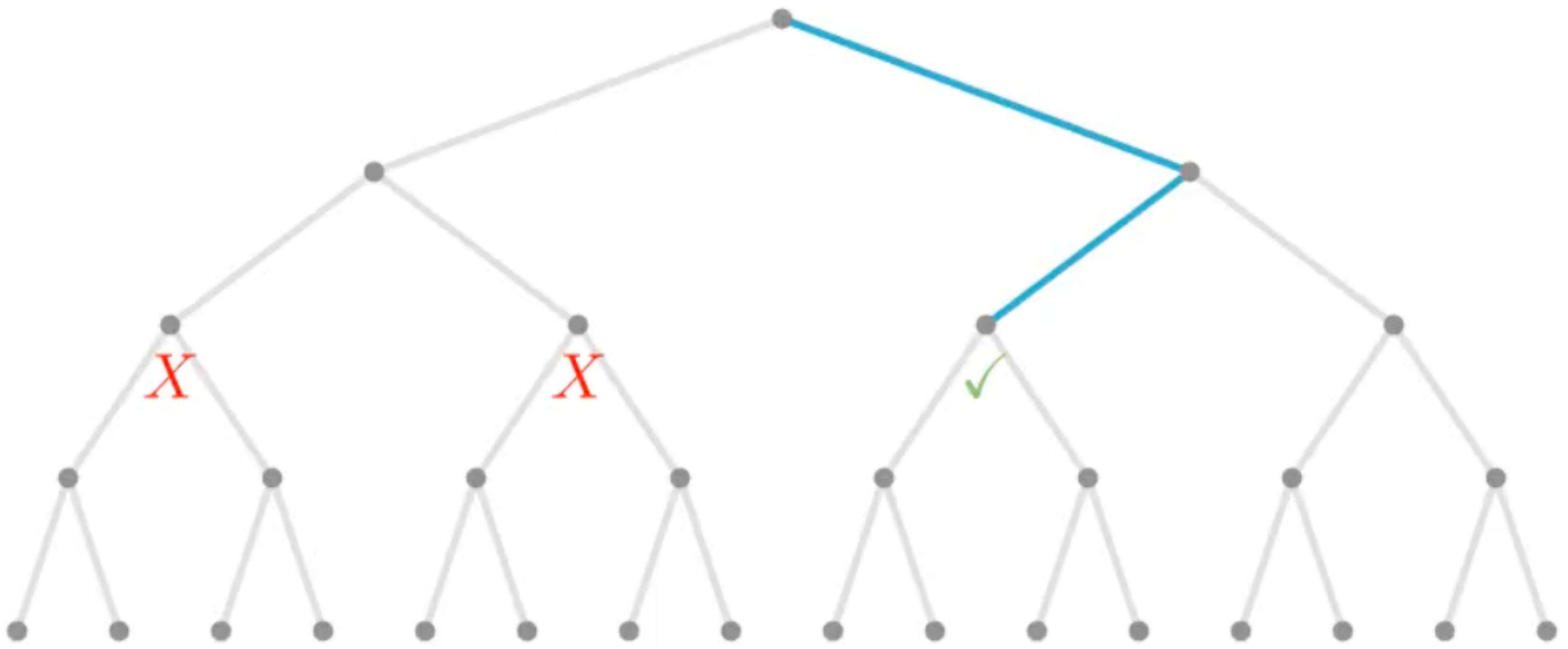

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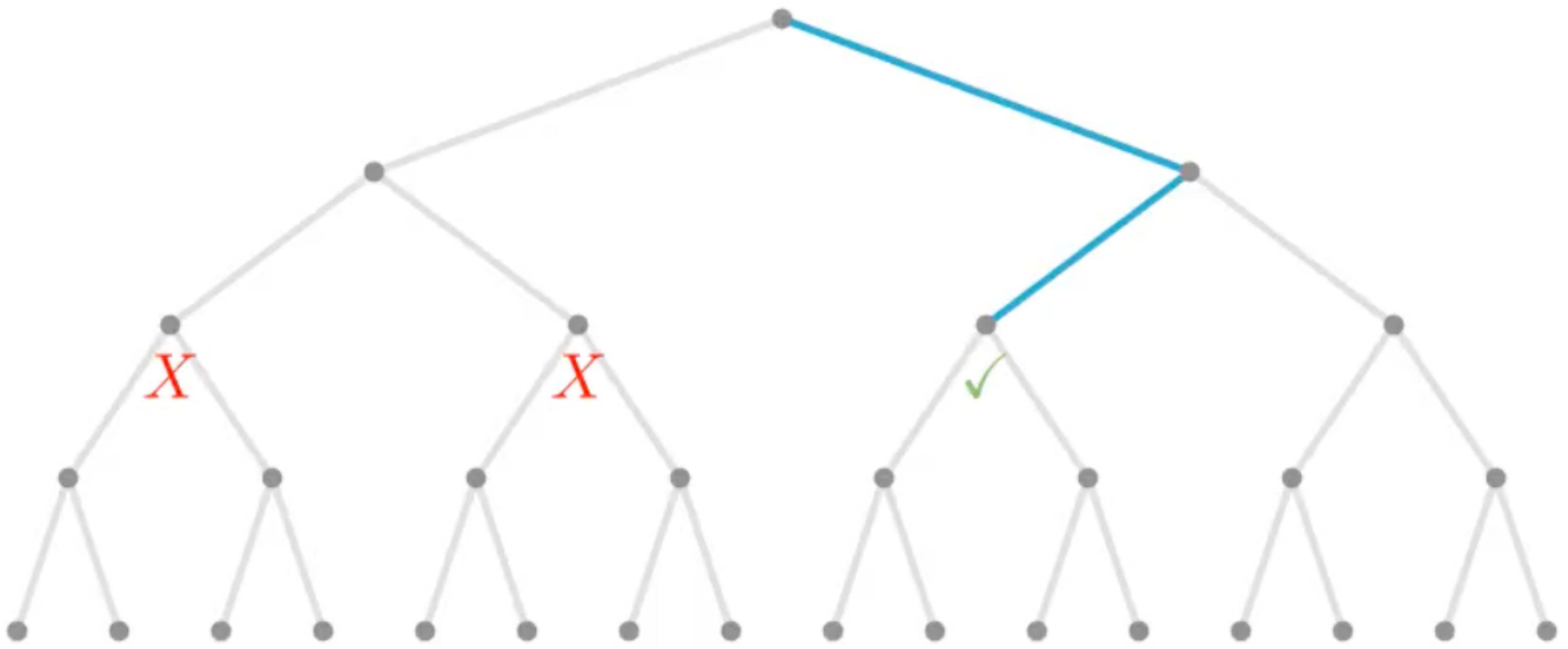

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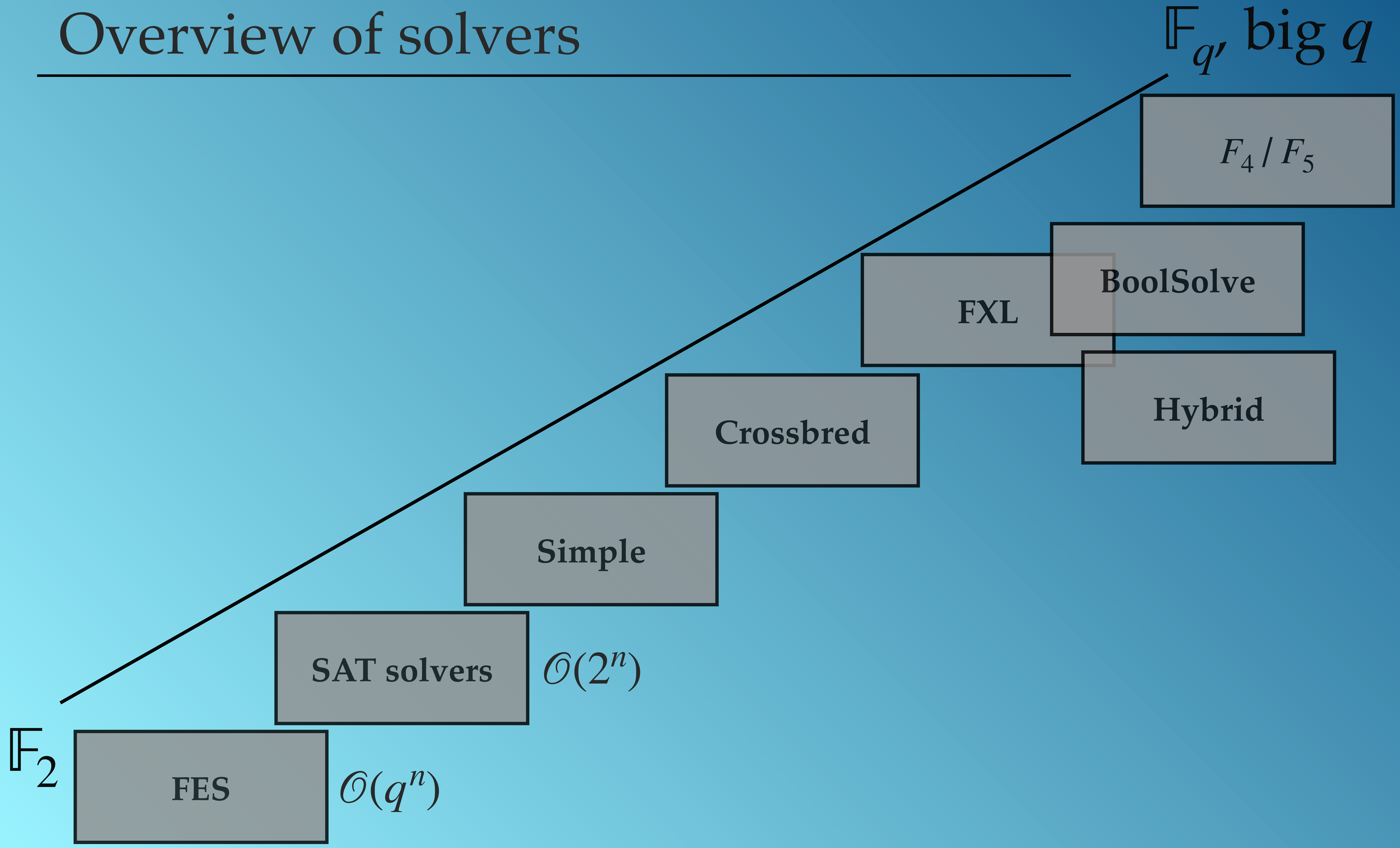
$$1 \cdot x_4 + 0 \cdot x_3 + 0 + x_3 + x_4 = 0$$

XOR-enabled SAT solvers: take as input XOR constraints as well; perform Gaussian elimination;
*CryptoMiniSat, WDSat

Overview of solvers



Macaulay matrix

Linearisation

Linear systems are easy to solve, nonlinear systems are hard.

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Example.

$$
\begin{aligned}
f_1 &: x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0\\
f_2 &: x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0\\
f_3 &: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0\\
f_4 &: x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0\\
f_5 &: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
\longrightarrow
\begin{aligned}
f_1 &: y_2 + y_5 + x_1 + x_3 + x_4 = 0\\
f_2 &: y_4 + y_3 + y_6 + x_1 + x_2 + x_4 = 0\\
f_3 &: y_5 + y_6 + x_1 + x_3 + 1 = 0\\
f_4 &: y_1 + y_2 + y_4 + x_3 + x_4 + 1 = 0\\
f_5 &: y_1 + y_4 + y_3 + x_3 = 0\\
f_6 &: y_2 + y_3 + y_6 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
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f_3 &: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0\\
f_4 &: \boxed{x_1x_2} + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0\\
f_5 &: \boxed{x_1x_2} + x_2x_3 + x_1x_4 + x_3 = 0\\
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\end{aligned}
\longrightarrow
\begin{aligned}
f_1 &: y_2 + y_5 + x_1 + x_3 + x_4 = 0\\
f_2 &: y_4 + y_3 + y_6 + x_1 + x_2 + x_4 = 0\\
f_3 &: y_5 + y_6 + x_1 + x_3 + 1 = 0\\
f_4 &: \boxed{y_1} + y_2 + y_4 + x_3 + x_4 + 1 = 0\\
f_5 &: \boxed{y_1} + y_4 + y_3 + x_3 = 0\\
f_6 &: y_2 + y_3 + y_6 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}$$

Linearisation

Linearisation adds solutions: a *random* quadratic system of m equations in n variables, when $n = m$, is expected to have one solution (probability is $\sim \frac{1}{q}$ for systems over $\mathbb{F}_q$). The corresponding linearised system has a solution space of dimension $\binom{n+1}{2} - m$.

$\binom{n}{2}$ quadratic plus n linear monomials

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$\binom{n}{2}$ quadratic plus n linear monomials

Loss of information: e.g. assignment $x_1 = 1; x_2 = 0; y_1 = 1;$ is part of a valid solution to the linearised system, but $x_1 x_2 \neq y_1$.

Macaulay matrix

Monomials →

Equations ↓

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1											
f_2											
f_3											
f_4											
f_5											
f_6											

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$
$$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$$
$$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$$
$$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$$
$$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$$
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Macaulay matrix

Monomials

Equations

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

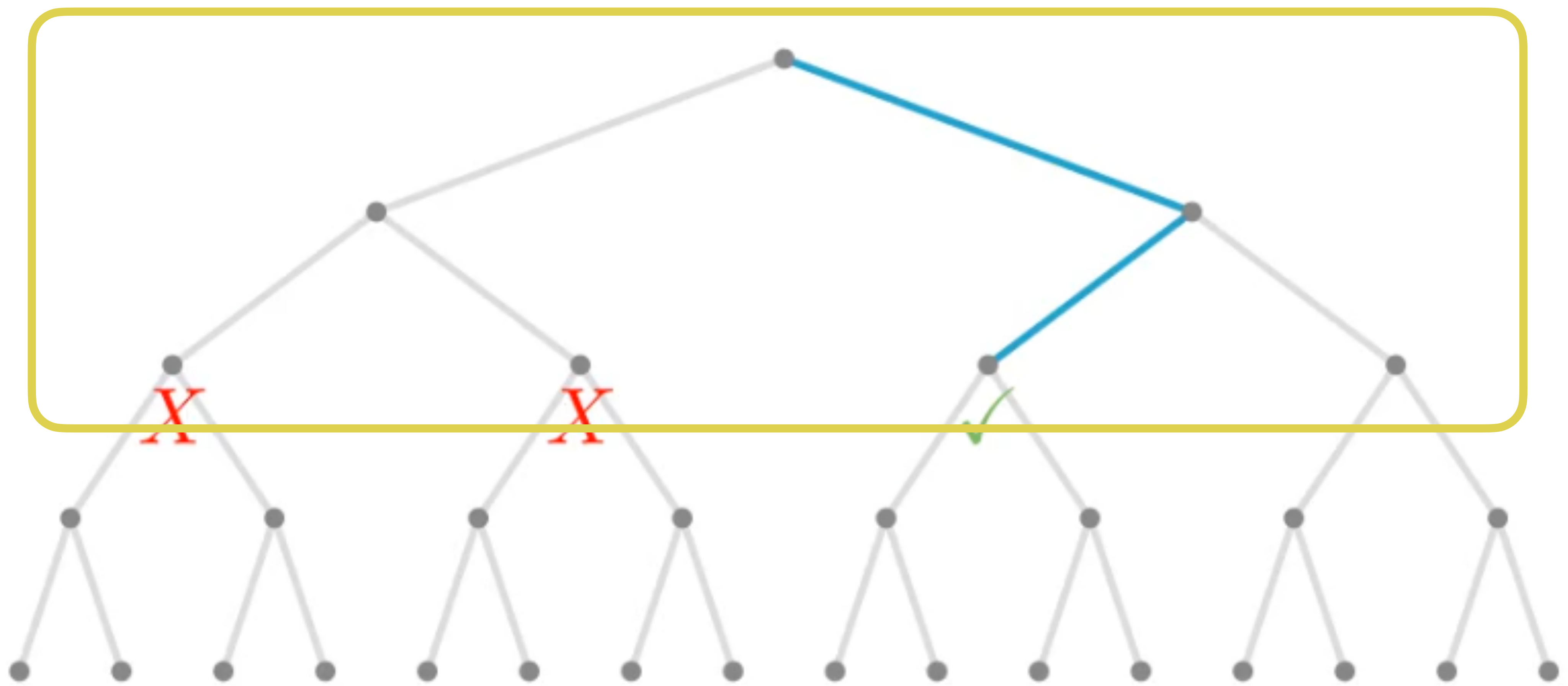
$$\begin{aligned}
f_1 &: x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0 \\
f_2 &: x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0 \\
f_3 &: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0 \\
f_4 &: x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0 \\
f_5 &: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0 \\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}$$

Simple algorithm

[Bouillaguet, Delaplace, T., 2021]

Simple algorithm

- Partial assignment
  - Gaussian elimination



$$1 \cdot 0 + 1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$
$$0 \cdot x_3 + 0 \cdot x_4 + 1 + 0 + 1 = 0$$
$$1 \cdot 0 + 0 \cdot x_3 + 0 \cdot x_4 + 1 + x_4 = 0$$
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Simple algorithm

Guess sufficiently many variables so that the remaining polynomial system can be solved by linearization.

Simple algorithm: complexity

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- n - number of variables
- m - number of equations

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Enumeration ends when:

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$$\mathcal{O}(2^{n-\sqrt{2m}})$$

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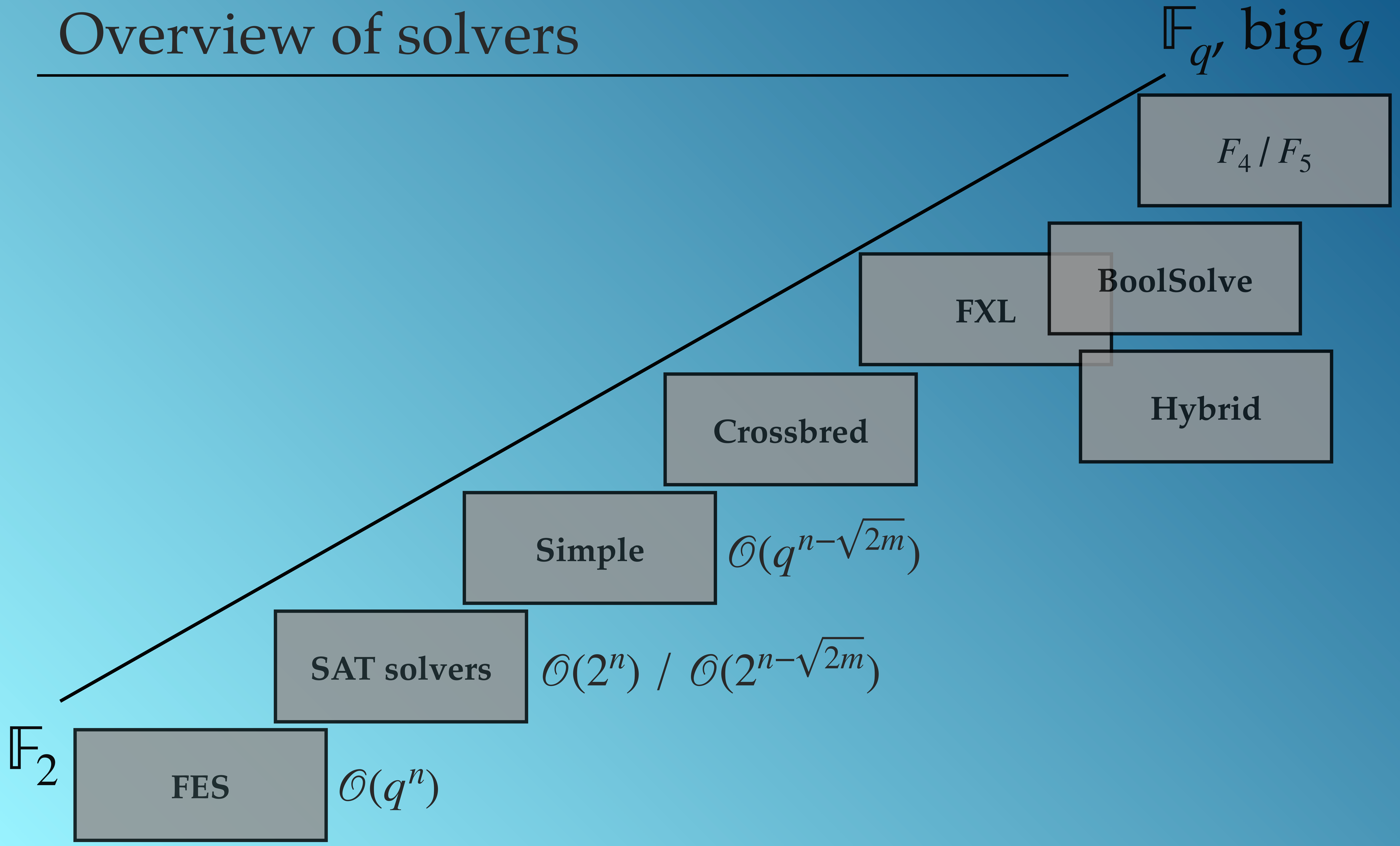
See also: Quantum BDT [Edme, Fouque, Schrottenloher]

Overview of solvers



Gröbner basis algorithms

[Buchberger, 1965]

[Lazard, 1983]

F_4/F_5 [Faugère, 1999/2002]

(XL [Courtois, Klimov, Patarin, Shamir, 2000])

Gröbner basis algorithms (intuition)

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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

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$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

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$\boxed{D = 3}$

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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0				
f_2	0	0	1	1	1	0	1	1	0	1	0				
f_3	0	0	0	1	0	1	0	1	1	0	1				
f_4	1	1	0	1	1	0	0	0	1	1	1				
f_5	1	0	1	1	1	0	0	0	1	0	0				
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x_1f_1															
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$\dots$															

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$\boxed{D = 4}$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
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$\dots$																
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- By the Nullstellensatz: $\mathbf{I}(V(I)) = I$, where $\mathbf{I}(V)$ denotes the ideal of V, i.e. $\mathbf{I}(V) = \{f \in R \,|\, f(a) = 0 \text{ for all } a \in V\}$ (Similar to Gauss' fundamental theorem, but for polynomials in many variables).

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Example. The shape of a GB with respect to the lexicographic order

$$
\begin{aligned}
f_1 &: x_1x_3 + x_1 + x_2x_4 + x_5 + x_6 + 1 = 0 \\
f_2 &: x_1x_4 + x_1 + x_2x_3 + x_2 + x_3x_4 + x_3x_6 + x_4 + x_5 = 0 \\
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\end{aligned}
\longrightarrow
\begin{aligned}
f_1' &: x_1 + x_6 = 0\\
f_2' &: x_2 + x_6 = 0\\
f_3' &: x_3 + x_6 = 0\\
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f_5' &: x_5 = 0
\end{aligned}
$$

*****
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*

$$V(< f_1, \ldots, f_6 >) = \{(0,0,0,1,0,0), (1,1,1,0,0,1)\}$$

Gröbner basis algorithms:

Buchberger, Lazard, F4, F5

Follow the core idea that we described, but combine the equations in an organised way, rather than multiplying them by all possible monomials.

Not covered in this talk:

- Monomial orders
- S-polynomials
- Polynomial long division
- Row reduction in parallel
- Reductions to zero
- Syzygy criterion
- …

XL / Gröbner basis algorithms: complexity

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$$\mathcal{O}\left(mD_{reg}\binom{n+D_{reg}-1}{D_{reg}}^{\omega}\right)$$

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D_{reg}: degree of regularity

the power of the first non-positive coefficient in the expansion of $$\frac{(1-t^2)^m}{(1-t)^n}$$

XL / Gröbner basis algorithms: complexity

```
m=8
n=7
R.<t> = PowerSeriesRing(ZZ)
hs = ((1-t^2)^(m)) / (1-t)^(n)
print(hs)
```

[3] ✓ 0.0s

```
1 + 7*t + 20*t^2 + 28*t^3 + 14*t^4 - 14*t^5 - 28*t^6 - 20*t^7 - 7*t^8 - t^9 + O(t^20)
```

The number of monomials (columns) minus linearly independent equations (rows) at degree $D = 4$ is 14.

XL / Gröbner basis algorithms: complexity

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Overview of solvers

$\mathbb{F}_q$, big q

F_4 / F_5 $\mathcal{O}(\binom{n+D_{reg}-1}{D_{reg}}^{\omega})$

BoolSolve

FXL

Hybrid

Crossbred

Simple $\mathcal{O}(q^{n-\sqrt{2m}})$

SAT solvers $\mathcal{O}(2^n)$ / $\mathcal{O}(2^{n-\sqrt{2m}})$

$\mathbb{F}_2$

FES $\mathcal{O}(q^n)$

FXL

[Courtois, Klimov, Patarin, Shamir, 2000]

Hybrid

[Bettale, Faugère, Perret, 2009]

BoolSolve

[Bardet, Faugère, Salvy, Spaenlehauer, 2013]

FXL, Hybrid, BoolSolve

Techniques are already covered in the previous section.

Algorithms will be explained in the summary.

The crossbred algorithm

[Joux, Vitse, 2017]

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
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\end{aligned}}$$

→ Put matrix in reduced row echelon form

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
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→ Take linear subsystem

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
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…

…if we had another 4 equations

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	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

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$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Subsystem is linear in variables $\{x_1, x_2, x_3\}$.

→ Enumerating x_4 will result in a linear subsystem.

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$

$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$

$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$

$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$

$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$

$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
…											

Crossbred algorithm

$f_1: x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2: x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4: x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5: x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Subsystem can be linearised

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Subsystem can be linearised

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
…											

…if we had another 4 equations, the subsystem would have a unique solution.

Otherwise: check candidate solutions against the other equations.

Crossbred algorithm

Parameters of the algorithm: D, k, d, h

- Enumerate h variables.
- Choose k of the remaining variables.
- Augment system up to degree D (compute degree-D Macaulay matrix).
- Take the subsystem that is at most degree d in the k chosen variables.
- Enumerate all but the k chosen variables.
- Linearise the subsystem and solve it.
- Check if candidate solutions are consistent with the rest of the system.

Crossbred algorithm

Parameters of the algorithm: D, k, d, h

- → Enumerate h variables.
- → Choose k of the remaining variables.
- → Augment system up to degree D (compute degree-D Macaulay matrix).
- → Take the subsystem that is at most degree d in the k chosen variables.
- → Enumerate all but the k chosen variables.
- → Linearise the subsystem and solve it.
- → Check if candidate solutions are consistent with the rest of the system.

The complexity is calculated as the best trade-off between the four parameters.

Crossbred algorithm

	Number of Variables (n)	Seed (0,1,2,3,4)	Date	Contestants	Computational Resource	Data
1	83	0	2023/09/16	Charles Bouillaguet and Julia Sauvage	https://gitlab.lip6.fr/almasty/hpXbred, 3488 AMD EPYC 7J13 cores on the Oracle public cloud	Details
6	74	0	2016/12/17	Antoine Joux	New hybridized XL related algorithm, Heterogeneous cluster of Intel Xeon @ 2.7-3.5 Ghz	Details
7	74	4	2017/11/15	Kai-Chun Ning, Ruben Niederhagen	Parallel Crossbred, 54 GPUs in the Saber cluster	Details
25	66	0	2016/01/22	Tung Chou, Ruben Niederhagen, Bo-Yin Yang	Gray Code enumeration, Rivyera, 128 Spartan 6 FPGAs	Details

Fukuoka MQ challenge record computations ($m = 2n$)

Overview of solvers

$\mathbb{F}_q$, big q

F_4 / F_5 $\mathcal{O}\left(\binom{n + D_{reg} - 1}{D_{reg}}^{\omega}\right)$

BoolSolve

FXL

Hybrid

Crossbred $\mathcal{O}(\ldots)$

Simple $\mathcal{O}(q^{n-\sqrt{2m}})$

SAT solvers $\mathcal{O}(2^n)$ / $\mathcal{O}(2^{n-\sqrt{2m}})$

$\mathbb{F}_2$

FES $\mathcal{O}(q^n)$

Summary

(Partial) enumeration

Candidate solutions (subsystem)

Conflict search

Extending to higher degrees

Computing a Gröbner Basis

FES

Simple

FXL

F_4 / F_5

SAT solvers

Crossbred

BoolSolve

Hybrid

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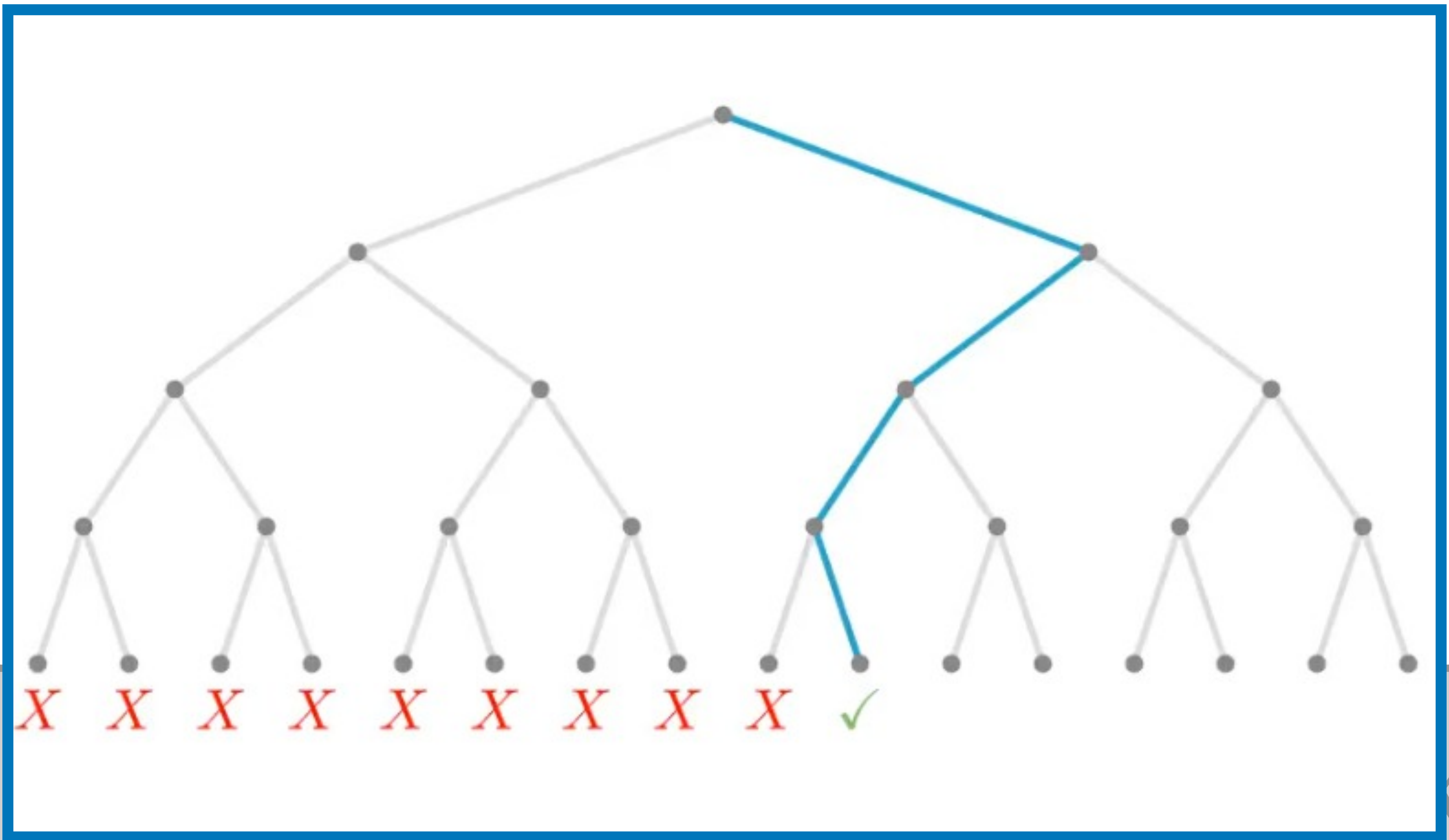

FES

XL

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	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
…											

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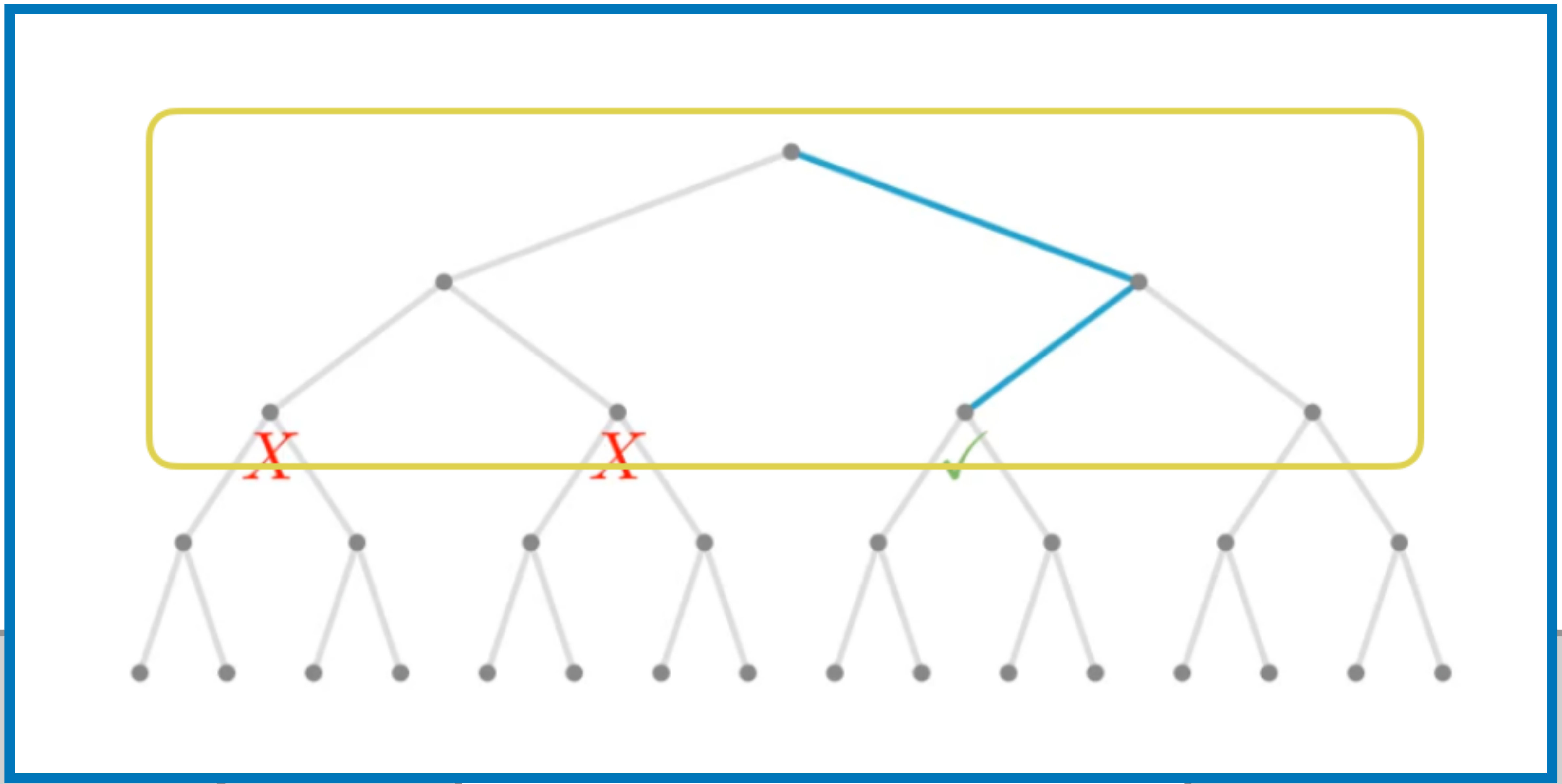

FES

F_4 / F_5

SAT solvers

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BoolSolve

Hybrid

Summary

(Partial) enumeration	Candidate solutions (subsystem)	Conflict search	Extending to higher degrees	Computing a Gröbner Basis

FES

Simple

FXL

F_4 / F_5

SAT solvers

Crossbred

BoolSolve

Hybrid

Summary

(Partial) enumeration

Candidate solutions

Conflict search

Extending to higher degrees

Computing a Gröbner Basis

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
x_2f_1																
$\dots$																
$x_1x_2f_1$																
$x_1x_3f_1$																

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$$f'_1 : x_1 + x_6 = 0$$
$$f'_2 : x_2 + x_6 = 0$$
$$f'_3 : x_3 + x_6 = 0$$
$$f'_4 : x_4 + x_6 + 1 = 0$$
$$f'_5 : x_5 = 0$$

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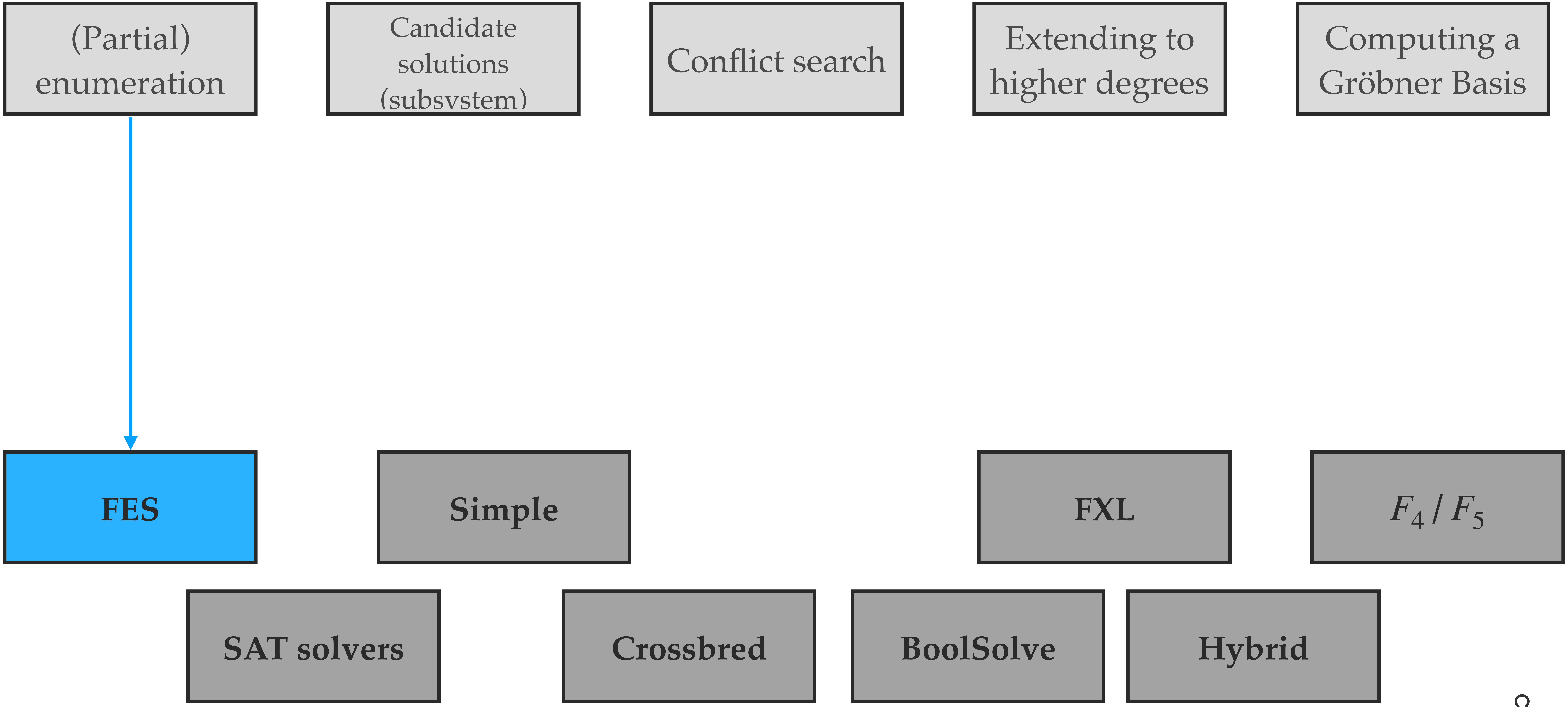
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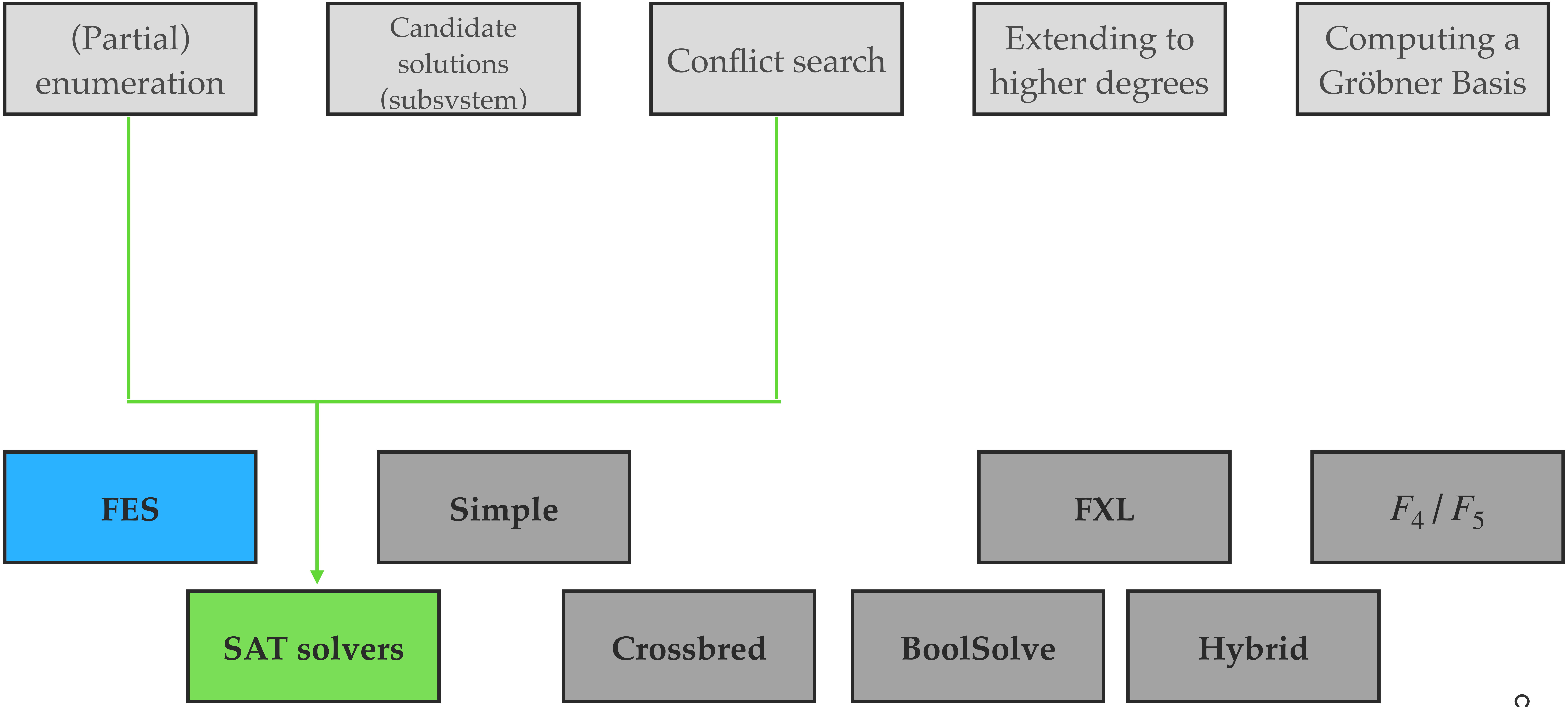
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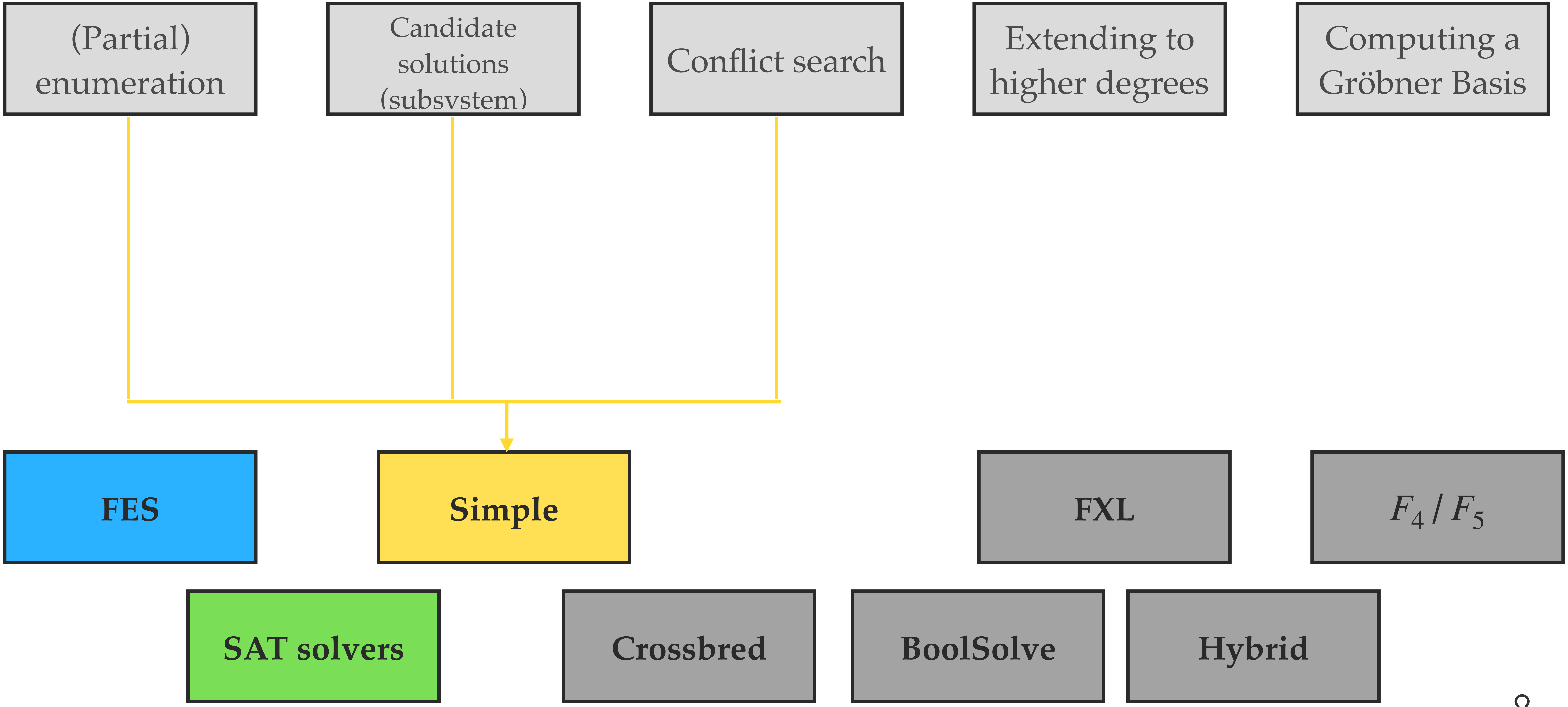
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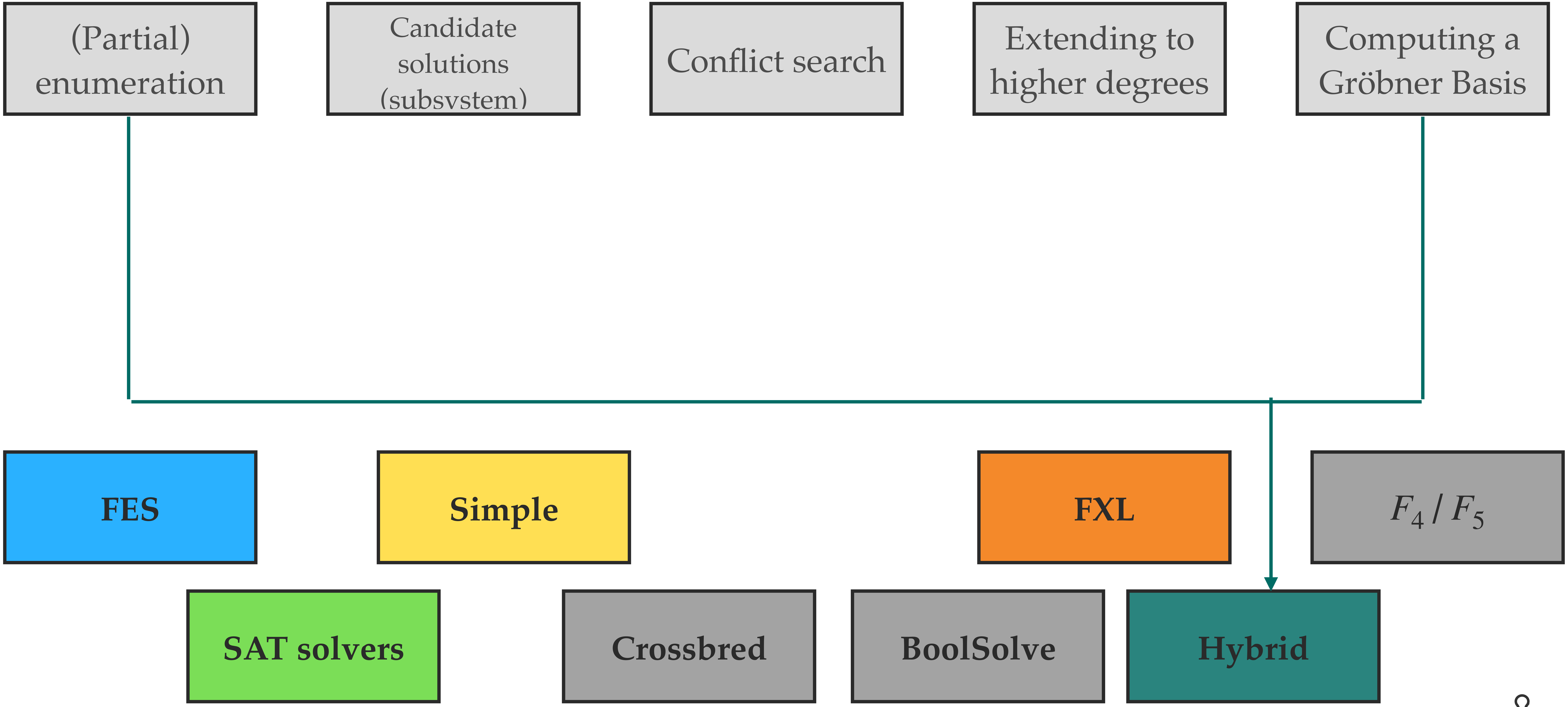
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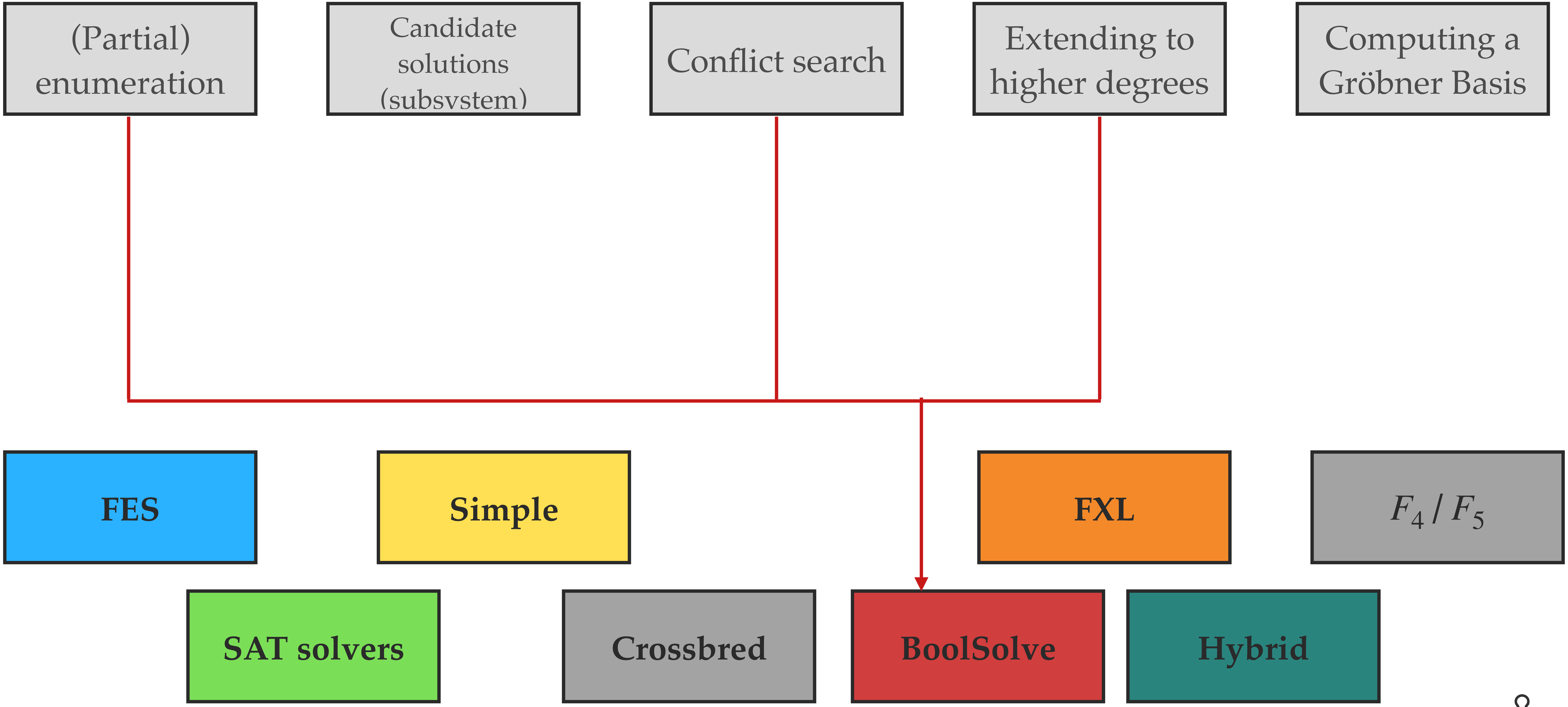

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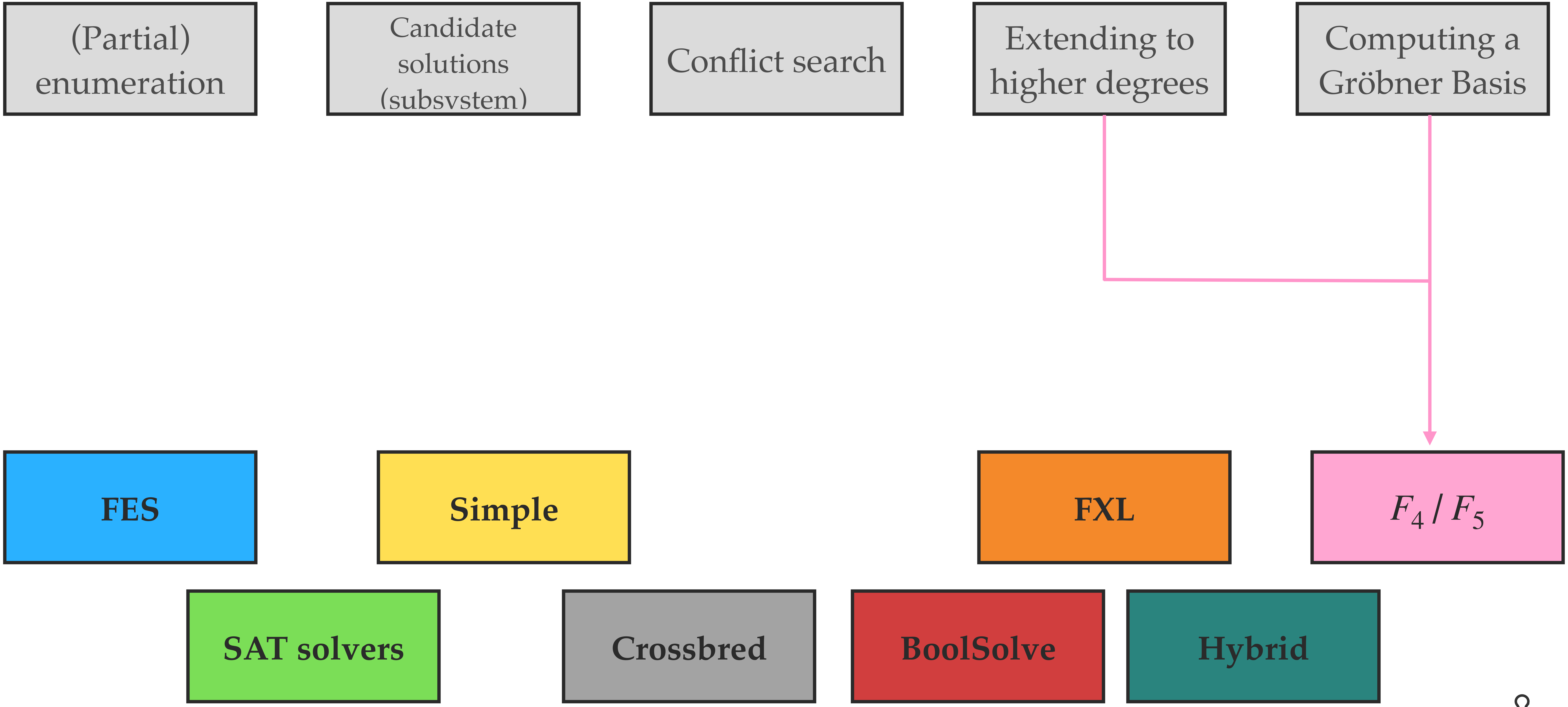
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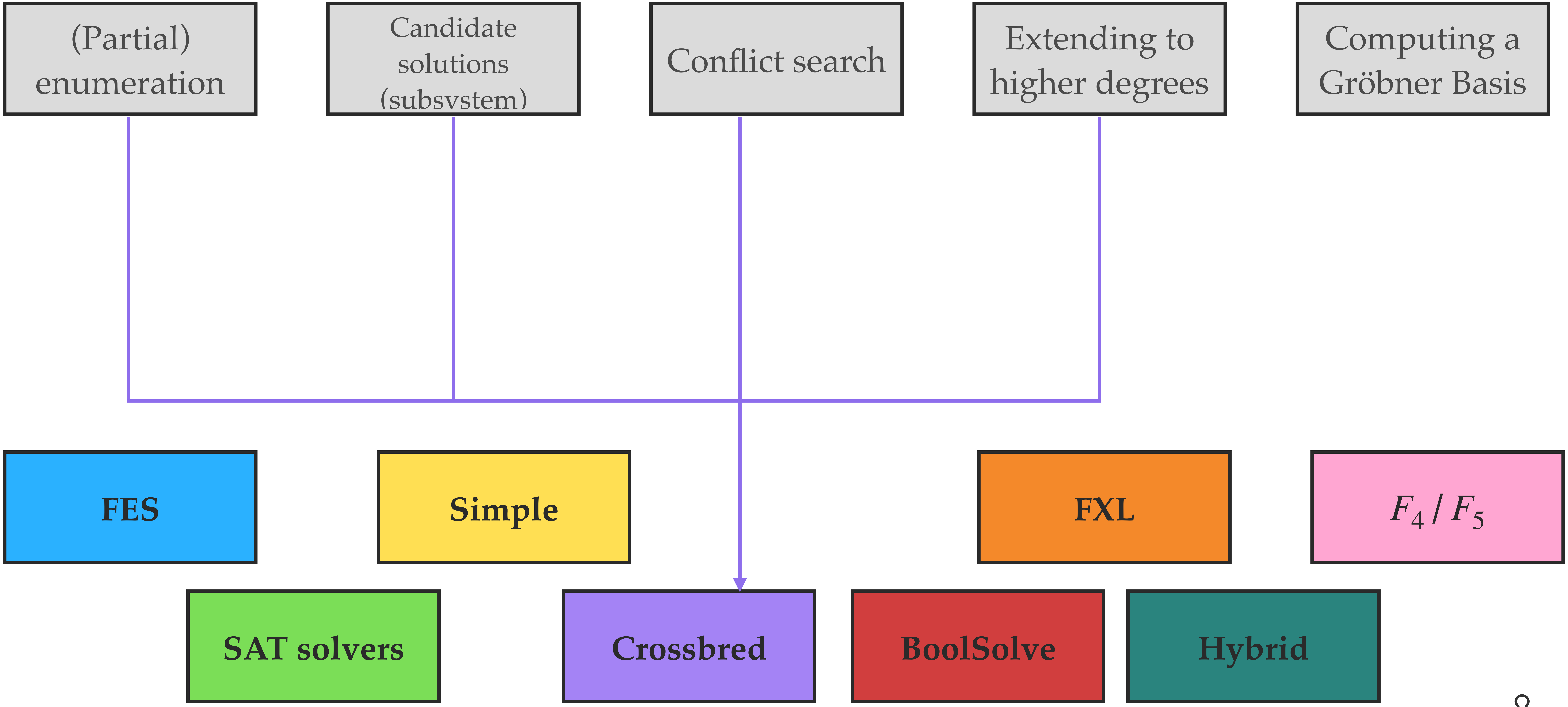

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Recap

- The MQ problem is (usually) hard.
- Modelisation can be crucial to how efficient an attack is.
- We have a variety of solvers for (over)determined systems.
- We can estimate the complexity of solving random systems, but for structured systems this requires deeper analysis.

To implement a solver and practice modelisation of different attacks:

Tutorial Tuesday, July 1

(install SageMath beforehand: https://github.com/LarsMath/tutorial-algebraic-cryptanalysis)

joint with Lars Ran

Credits: created with ♥, and Manim.